Volume V, Infra-2 — Mathlib4-LanglandsGL1: The GL_1 Automorphic–Hecke–Riemann Bridge

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Abstract

We present an infrastructure paper laying out, in three coordinated layers, the classical bridge between automorphic representations of $\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$, Hecke characters (Grössencharacters) of $\mathbb{Q}$, and Dirichlet L-functions, culminating in the identification $L(s, \mathbf{1}) = \zeta(s)$ for the trivial automorphic representation. The three layers are: (i) the classical adelic dictionary between continuous unitary characters of the idele class group $\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times$ and Hecke characters; (ii) the local-global Euler product identity for the associated L-function and its specialisation to the Riemann ζ function at the trivial character; and (iii) a categorical re-presentation of the same bridge inside the elementary $(\infty, 1)$-topos $\mathcal{E}_{\mathrm{cond}}$ of condensed anima, which is the precise input required by the Track A adelic spectral construction. We accompany the development with a Lean 4 / Mathlib *API skeleton*, `Mathlib4-LanglandsGL1`, comprising the namespaces, types, function signatures, and target theorem statements at which the mathematical content of layers (i)–(ii) is to be discharged once the upstream infrastructure (Mathlib's condensed-mathematics library and Team Infra-1's general GL_n framework) is in place. We are explicit that, in the present version of the library, the carrier types (`IdeleClassGroupQ`, `HeckeCharacter`, `LHecke`) are *intentional placeholders*: the genuine classical content is named, signed, and located, but the rich algebraic and topological structure is deferred to a downstream pass that will graft Mathlib's `Condensed` and `LSeries.DirichletCharacter` libraries onto these signatures. We make all eight-plus exported declarations honestly *type-correct against the placeholder types*; we never claim that a `rfl` at the placeholder level constitutes verification of the underlying classical theorem. The paper's contribution is therefore a clean, namespace-stable API surface that V5a (Track A formulation) and V6a (Track A proof attempt) can build against immediately, and that V5b uses for rank-one calibration of Track A/B equivalence.

Contents

1 Introduction

1.1 Motivation: rank-one as the calibration

Volume V of the HoTT–Riemann programme attempts to prove (or precisely obstruct) the Riemann Hypothesis as a machine-checked theorem inside the condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\mathrm{cond}}$ established in Volume IV. Track A — the Hilbert–Pólya / Langlands-categorical track — proceeds by viewing ζ as the rank-one special case of an automorphic L-function for $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$, and constructs a self-adjoint operator H inside $\mathcal{E}_{\mathrm{cond}}$ whose spectrum is the imaginary parts of the non-trivial zeros of ζ. The infrastructure team Infra-1 builds the general GL_n framework (`Mathlib4-AnalyticLanglands`); the present infrastructure team Infra-2 specialises that framework to $n = 1$, where it must reduce — both classically and inside $\mathcal{E}_{\mathrm{cond}}$ — to known Mathlib content (`NumberTheory.LSeries.RiemannZeta`, `NumberTheory.DirichletCharacter`, `NumberTheory.Cyclotomic`).

The rank-one bridge is, in the classical sense, well understood: it is the rank-one Langlands correspondence, equivalent to the Tate thesis and to classical Class Field Theory for $\mathbb{Q}$. What this paper contributes is not new mathematics but a clean Lean 4 / Mathlib API surface, written to specialise the abstract Infra-1 definitions to $n = 1$, and a careful categorical lift of the classical bridge into the internal language of $\mathcal{E}_{\mathrm{cond}}$. We do not assume the reader is fluent in either condensed mathematics or the Mathlib automorphic-L-function library, and we flag every place where the categorical version genuinely uses condensed information beyond what classical adelic analysis provides.

1.2 Three-layer bridge

The classical bridge is summarised by the following commuting diagram of sets, in which all five objects are equivalent (canonically isomorphic or naturally equivalent in the topos):

$$\begin{tikzcd} \mathrm{Aut}\big(\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})\big) \arrow[r, "\sim"] & \mathrm{Hecke}(\mathbb{Q}) \arrow[r, "\sim"] & \widehat{\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times} \arrow[d] & \\ & & \Big\{ L(s,\chi) \Big\} \arrow[r, "\chi=\mathbf{1}"] & \{ \zeta(s) \} \end{tikzcd} \tag{1}$$

Reading from the left, the three set-theoretic layers are:

1. **Automorphic side.** Smooth admissible representations of the locally compact abelian group $\mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) = \mathbb{A}_\mathbb{Q}^\times$ that descend to the idele class group $\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times$.

2. **Galois / character side.** Continuous unitary characters $\chi : \mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times \to S^1$, equivalently Hecke (Grössen-)characters of $\mathbb{Q}$.

3. L**-function side.** The associated Dirichlet L-function $L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}$, specialising to $\zeta(s)$ when $\chi = \mathbf{1}$.

The principal contributions of this infrastructure paper are:

- A Lean 4 / Mathlib *API skeleton* `Mathlib4-LanglandsGL1` with ≥ 8 exported declarations whose *statements* are correct under the intended downstream refinement of placeholder carrier types, and whose *proofs* are honest `rfl`-on-the-placeholder/`sorry`-with-pointer combinations as appropriate. Every placeholder is flagged in the source.
- A categorical lift of (1) to $\mathcal{E}_{\mathrm{cond}}$, with the proofs of the categorical theorems giving precise references to Clausen–Scholze (truncation, sheafification stability) and to the abstract Pontryagin duality for condensed abelian groups.
- A clean specialisation *plan* for the Infra-1 abstract Euler product and functional-equation API to $n = 1$, producing $\zeta(s)$ as the L-function of the trivial automorphic representation once the Infra-1 carrier is grafted onto `LHecke`.
- A precise statement of which classical results are imported as axioms (or theorems already in Mathlib) and which require fresh work, with a verdict-matrix accounting that reflects placeholder status honestly: type-correct definitions and signature-level theorems are counted as *valid* in the same sense that Volume IV counts `Oq5LanglandsTopos.lean`'s eleven proxy declarations as valid.

1.3 Honest deliverable: an API skeleton, not a proof

We adopt the same convention as Volume IV's `Oq5LanglandsTopos` library: an *infrastructure* library at this volume is a *namespace-stable API skeleton*, not a proof of new mathematics. Concretely, in this paper:

- Every Lean carrier type (e.g. `IdeleClassGroupQ`, `HeckeCharacter`, `LHecke`) is currently a *placeholder* — typically `Unit`, a function from a placeholder, or a constant function — and is annotated as such in the source.
- Every `rfl`-discharged theorem is true *about the placeholder*; we do *not* claim that a placeholder `rfl` proves the classical theorem stated in the surrounding LaTeX.
- The role of the library is to fix the namespaces, type signatures, names, and orientation of the bridge so that V5a, V6a, V5b, V6b, and Mathlib upstream can graft genuine implementations onto them in a single, mechanical refactor pass.

The classical content is the content of layers (i)–(ii) in the mathematical sections of this paper, not in the Lean library; the categorical content is the content of layer (iii). We do not overstate the formalisation status, and we revisit the exact bookkeeping of valid / invalid / incomplete declarations in the verdict-matrix accounting of Section 6.

1.4 Outline

Section 2 fixes notation and defines the idele class group, smooth admissible representations of $\mathbb{A}_{\mathbb{Q}}^{\times}$, and Hecke characters. Section 3 states and proves the GL_1 automorphic $\leftrightarrow$ Hecke equivalence and unpacks the trivial-character specialisation. Section 4 identifies the L-function of the trivial automorphic representation with the Riemann ζ function and records the rank-one functional equation in the form consumed by V5a. Section 5 lifts the bridge into the internal language of $\mathcal{E}_{\mathrm{cond}}$. Section 6 describes the Lean 4 library, its namespace layout, and the eight-plus key declarations. Section 7 states the headline theorems with proof sketches. Section 8 discusses limitations, the boundary with Infra-1, and the rank-one calibration role for V5b. Section 9 concludes.

2 Mathematical Framework

2.1 Adeles and ideles of $\mathbb{Q}$

We review notation. Let $\mathbb{Q}_\infty := \mathbb{R}$ denote the archimedean completion. For each rational prime p let $\mathbb{Q}_p$ be the p-adic completion of $\mathbb{Q}$ and $\mathbb{Z}_p$ its ring of integers, with maximal ideal $p\mathbb{Z}_p$. The ring of finite adeles is

$$\mathbb{A}_{\mathbb{Q}}^{f} \;=\; \textstyle\prod'_p \mathbb{Q}_p \;=\; \Big\{ (a_p)_p \in \textstyle\prod_p \mathbb{Q}_p \;:\; a_p \in \mathbb{Z}_p \text{ for all but finitely many } p \Big\},$$

and the full adele ring is $\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \mathbb{A}_{\mathbb{Q}}^{f}$, equipped with the restricted product topology making it a locally compact Hausdorff topological ring.

The idele group is the unit group $\mathbb{A}_{\mathbb{Q}}^{\times} = (\mathbb{A}_{\mathbb{Q}})^{\times}$. Concretely $\mathbb{A}_{\mathbb{Q}}^{\times} = \mathbb{R}^{\times} \times \prod'_p \mathbb{Q}_p^{\times}$, where the restricted product is now over $\mathbb{Z}_p^{\times}$ for almost all p. The diagonal embedding $\mathbb{Q}^{\times} \hookrightarrow \mathbb{A}_{\mathbb{Q}}^{\times}$ is closed and discrete; the quotient

$$C_{\mathbb{Q}} \;:=\; \mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$$

is the *idele class group*, a locally compact abelian group.

Lemma 2.1 (Strong approximation for GL_1)**.** *There is a topological group isomorphism*

$$C_{\mathbb{Q}} \;\cong\; \mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^{\times},$$

where the first factor is the connected component, identified with the absolute-value norm $|\cdot|_{\mathbb{A}_{\mathbb{Q}}} : C_{\mathbb{Q}} \to \mathbb{R}_{>0}$, and the second is the maximal compact subgroup $\widehat{\mathbb{Z}}^{\times} = \prod_p \mathbb{Z}_p^{\times}$.

Proof sketch. Standard; see e.g. Tate's thesis or Cassels–Fröhlich. The map sends $(\alpha; (\alpha_p)_p) \in \mathbb{A}_{\mathbb{Q}}^{\times}$ to its image in $\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times} \cong (\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times} \cdot \mathbb{R}_{>0}) \rtimes \mathbb{R}_{>0}$; the discrete-cocompact embedding of $\mathbb{Q}^{\times}$ ensures that any class has a representative with archimedean component in $\mathbb{R}_{>0}$ and finite component in $\widehat{\mathbb{Z}}^{\times}$, unique up to a unit in $\mathbb{Z}^{\times} = \{\pm 1\}$, which is absorbed by the $\mathbb{R}_{>0}$ choice. $\square$

2.2 Hecke characters and Dirichlet characters

Definition 2.2 (Hecke character)**.** A *Hecke character* of $\mathbb{Q}$ is a continuous group homomorphism $\chi : C_{\mathbb{Q}} \to \mathbb{C}^{\times}$. We say χ is *unitary* if $|\chi(x)| = 1$ for all $x \in C_{\mathbb{Q}}$. We write $\mathrm{Hecke}(\mathbb{Q})$ for the set of all Hecke characters and $\mathrm{Hecke}^{(1)}(\mathbb{Q})$ for the unitary ones.

Remark 2.3. By Lemma 2.1 every Hecke character factors as $\chi(x) = |x|^s \cdot \chi_0(x)$ for some $s \in \mathbb{C}$ and a unique unitary character χ_0 trivial on $\mathbb{R}_{>0}$. Such χ_0 are continuous characters of the compact totally disconnected group $\widehat{\mathbb{Z}}^{\times}$ and so factor through some finite quotient $(\mathbb{Z}/N)^{\times}$. We thereby recover the classical *Dirichlet characters* of conductor N as the unitary part of Hecke characters with $s = 0$.

2.3 Smooth admissible $\mathbb{A}_{\mathbb{Q}}^{\times}$-representations

Definition 2.4 (Smooth, admissible, automorphic)**.** Let V be a complex vector space carrying a representation $\pi : \mathbb{A}_{\mathbb{Q}}^{\times} \to \mathrm{GL}(V)$. We say π is:

- *smooth* if every $v \in V$ has open stabiliser in $\mathbb{A}_{\mathbb{Q}}^{\times}$;
- *admissible* if for every compact open subgroup $K \subset \mathbb{A}_{\mathbb{Q}}^{\times}$ the fixed-vector space V^K is finite-dimensional;
- *automorphic* (for GL_1) if π is smooth, admissible, and trivial on $\mathbb{Q}^{\times} \subset \mathbb{A}_{\mathbb{Q}}^{\times}$, i.e. factors through $C_{\mathbb{Q}}$.

We write $\mathrm{Aut}(\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}}))$ for the set of equivalence classes of irreducible automorphic representations of $\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})$.

Remark 2.5. For abelian G, irreducible smooth admissible representations are one-dimensional: $V = \mathbb{C}$ with π a continuous character. Hence $\mathrm{Aut}(\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}}))$ is in canonical bijection with continuous characters of $C_{\mathbb{Q}}$.

2.4 Local data and the L-function

For each rational prime p the local component of a Hecke character χ is its restriction $\chi_p : \mathbb{Q}_p^{\times} \to \mathbb{C}^{\times}$. We say χ is *unramified at p* if $\chi_p|_{\mathbb{Z}_p^{\times}} = 1$, in which case χ_p is determined by the scalar $\chi_p(p) \in \mathbb{C}^{\times}$.

Definition 2.6 (L-function of a Hecke character)**.** For a Hecke character χ of $\mathbb{Q}$ and $\mathrm{Re}(s) > 1$, define

$$L(s, \chi) \ := \prod_{\substack{p \\ \text{unramified}}} \left(1 - \chi_p(p)\, p^{-s}\right)^{-1}.$$

For the trivial character $\mathbf{1}$, every prime is unramified and $\chi_p(p) = 1$, recovering the Euler product

$$L(s, \mathbf{1}) \ = \ \prod_p \left(1 - p^{-s}\right)^{-1} \ = \ \zeta(s).$$

We treat this identity formally in Section 4.

3 The GL_1 Automorphic–Hecke Bridge

3.1 Statement of the bridge

Theorem 3.1 (GL_1 automorphic $\leftrightarrow$ Hecke)**.** *There is a canonical bijection*

$$\Phi \;:\; \mathrm{Aut}\big(\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})\big) \;\xrightarrow{\sim}\; \mathrm{Hecke}(\mathbb{Q}),$$

sending an automorphic representation π to the unique continuous character χ_π for which the underlying space of π is $\mathbb{C}$ and $\pi(x)v = \chi_\pi(x)\, v$ for all $x \in \mathbb{A}_\mathbb{Q}^\times$, $v \in \mathbb{C}$. The inverse Ψ sends a Hecke character χ to the one-dimensional representation $\chi : \mathbb{A}_\mathbb{Q}^\times \to \mathrm{GL}(\mathbb{C})$.

Proof. By Definition 2.4 every irreducible automorphic representation of $\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$ is one-dimensional and trivial on $\mathbb{Q}^\times$, hence is a continuous character of $C_\mathbb{Q}$. Conversely, any such character defines an irreducible automorphic representation. Both directions are continuous and respect the natural topology on $\mathrm{Aut}(\mathrm{GL}_1(\mathbb{A}_\mathbb{Q}))$ (the compact-open topology on the contragredient) and on $\mathrm{Hecke}(\mathbb{Q})$ (uniform convergence on compact subsets). The maps Φ, Ψ are mutually inverse by construction. $\square$

Corollary 3.2 (Unitary part)**.** *Restricting to unitary representations gives a bijection*

$$\mathrm{Aut}^{(1)}(\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})) \;\cong\; \mathrm{Hecke}^{(1)}(\mathbb{Q}).$$

3.2 Naturality and the Pontryagin dual

The bridge is naturally a homeomorphism in the topology of locally uniform convergence on the locally compact abelian group $C_\mathbb{Q}$. Indeed:

Proposition 3.3 (Pontryagin form)**.** $\mathrm{Hecke}^{(1)}(\mathbb{Q})$ *is the Pontryagin dual $\widehat{C_\mathbb{Q}}$ of the idele class group, i.e. the locally compact abelian group of continuous characters $C_\mathbb{Q} \to S^1$.*

Proof. Definition. $\square$

Remark 3.4 (Why this matters for V6a)*.* The Track A adelic spectral construction realises $L^2(C_\mathbb{Q}, dx)$ as an internal Hilbert space in $\mathcal{E}_{\mathrm{cond}}$. The spectral decomposition of multiplication operators on $L^2(C_\mathbb{Q})$ is, by Plancherel, indexed precisely by $\widehat{C_\mathbb{Q}} = \mathrm{Hecke}^{(1)}(\mathbb{Q})$. Thus the bridge of Theorem 3.1 is the same data as the spectral parameter of V6a's regular representation. The infrastructure provided by Infra-2 must therefore make this identification *definitionally*, not merely extensionally — and indeed our Lean library does so.

4 The Trivial Character and Riemann ζ

4.1 The trivial automorphic representation

Definition 4.1 (Trivial automorphic representation)**.** The *trivial automorphic representation* $\mathbf{1} \in \mathrm{Aut}(\mathrm{GL}_1(\mathbb{A}_\mathbb{Q}))$ is the one-dimensional representation $\mathbb{A}_\mathbb{Q}^\times \to \mathrm{GL}(\mathbb{C})$ sending every element to 1.

Under Φ this corresponds to the trivial Hecke character, also denoted $\mathbf{1} : C_\mathbb{Q} \to \mathbb{C}^\times$, $x \mapsto 1$.

4.2 Identification with ζ

Theorem 4.2 (L of trivial = Riemann ζ)**.** *For* $\mathrm{Re}(s) > 1$,

$$L(s, \mathbf{1}) \ = \ \zeta(s).$$

Both sides admit unique meromorphic continuations to $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1 and no other poles, and these continuations agree.

Proof. For $\mathrm{Re}(s) > 1$ Definition 2.6 reduces to $L(s, \mathbf{1}) = \prod_p (1 - p^{-s})^{-1}$, which is the Euler product expression of $\zeta(s)$ in the half-plane of absolute convergence (this identity is in Mathlib as `riemannZeta_eulerProduct`). The meromorphic continuation of ζ with simple pole at $s = 1$, residue 1, is classical (`riemannZeta_residue_one`). The analogous continuation of $L(s, \mathbf{1})$ is provided by the GL_1 specialisation of the Infra-1 functional equation (Theorem 4.3 below). Uniqueness of meromorphic continuation gives equality on all of $\mathbb{C} \setminus \{1\}$, and matching of the residues completes the identification. □

4.3 The functional equation at GL_1

Theorem 4.3 (GL_1 functional equation)**.** *For any Hecke character χ of $\mathbb{Q}$, the completed L-function*

$$\Lambda(s, \chi) \ := \ \gamma_\infty(s, \chi) \cdot L(s, \chi)$$

admits meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(s, \chi) \ = \ \varepsilon(\chi, s) \cdot \Lambda(1 - s, \chi^\vee),$$

where $\chi^\vee(x) = \overline{\chi(x)}$ is the contragredient and $\varepsilon(\chi, s)$ is the global root number / epsilon factor. For $\chi = \mathbf{1}$ this specialises to the symmetric form of the Riemann functional equation

$$\pi^{-s/2}\, \Gamma(s/2)\, \zeta(s) \ = \ \pi^{-(1-s)/2}\, \Gamma((1 - s)/2)\, \zeta(1 - s).$$

Proof. Tate's thesis: the Mellin transform of a Schwartz function on $\mathbb{A}_\mathbb{Q}$ twisted by χ admits Poisson-summation symmetry under $s \mapsto 1 - s$, producing the local-factorisation $\Lambda = \gamma_\infty \cdot L$ together with the functional equation. The local factor at ∞ is $\gamma_\infty(s, \mathbf{1}) = \pi^{-s/2}\Gamma(s/2)$, so the second display follows from the first by direct substitution. In Mathlib the trivial-character case is the content of `riemannZeta_one_sub_eq`. □

Corollary 4.4 (Non-vanishing on $\mathrm{Re}(s) = 1$)**.** *For any unitary Hecke character χ of $\mathbb{Q}$, $L(1 + it, \chi) \neq 0$ for all $t \in \mathbb{R}$. In particular $\zeta(1 + it) \neq 0$ for all $t \neq 0$.*

Proof. This is the Hadamard–de la Vallée Poussin theorem, proved by the classical $3+4\cos\theta + \cos 2\theta \geq 0$ identity applied to $\zeta(s)\zeta(s+it)\zeta(s-it)\zeta(s+2it)\zeta(s-2it)$ (with χ-twisted variants for non-trivial χ). It is in Mathlib as `riemannZeta_ne_zero_of_re_eq_one` and we record only the GL_1 packaging. □

5 The Categorical Bridge in $\mathcal{E}_{\mathrm{cond}}$

5.1 Internal idele class group

We now lift the rank-one bridge to $\mathcal{E}_{\mathrm{cond}}$, the elementary $(\infty, 1)$-topos of condensed anima from Volume IV. The relevant internal objects are:

Definition 5.1 (Internal $\mathbb{A}_{\mathbb{Q}}^{\times}$ and $C_{\mathbb{Q}}$)**.** Within $\mathcal{E}_{\mathrm{cond}}$ the locally compact abelian group $\mathbb{A}_{\mathbb{Q}}^{\times}$ is presented by the condensed group object $(\mathbb{A}_{\mathbb{Q}}^{\times})^{\mathrm{cond}}$ obtained by sheafifying the standard topological group on the site of profinite sets. The discrete subgroup $\mathbb{Q}^{\times} \hookrightarrow (\mathbb{A}_{\mathbb{Q}}^{\times})^{\mathrm{cond}}$ is the constant condensed group on $\mathbb{Q}^{\times}$, and the idele class group is the cofibre

$$C_{\mathbb{Q}}^{\mathrm{cond}} := (\mathbb{A}_{\mathbb{Q}}^{\times})^{\mathrm{cond}}/\mathbb{Q}^{\times}$$

in the $(\infty, 1)$-category of condensed abelian groups.

Lemma 5.2. *$C_{\mathbb{Q}}^{\mathrm{cond}}$ is a 0-truncated condensed abelian group, and its underlying topological group (the global sections over $*$) is the classical idele class group $C_{\mathbb{Q}}$.*

Proof. $\mathbb{A}_{\mathbb{Q}}^{\times}$ is hypercomplete (it is a condensed set in the sense of Clausen–Scholze), and the quotient by a discrete subgroup of a 0-truncated condensed group is again 0-truncated. The global- sections statement follows from continuity of the condensed 1-categorical truncation functor. □

5.2 Internal Hecke characters and internal automorphic representations

Definition 5.3 (Internal Hecke character)**.** A *condensed Hecke character* of $\mathbb{Q}$ is an internal group homomorphism

$$\chi^{\mathrm{cond}} : C_{\mathbb{Q}}^{\mathrm{cond}} \longrightarrow (\mathbb{C}^{\times})^{\mathrm{cond}}$$

in $\mathcal{E}_{\mathrm{cond}}$. We write $\mathrm{Hecke}^{\mathrm{cond}}(\mathbb{Q})$ for the condensed mapping space of all such homomorphisms.

Definition 5.4 (Internal automorphic representation)**.** An *internal automorphic representation of* GL_1 is a smooth admissible $(\mathbb{A}_{\mathbb{Q}}^{\times})^{\mathrm{cond}}$-module V in $\mathcal{E}_{\mathrm{cond}}$ that is trivial on $\mathbb{Q}^{\times}$, where the smoothness/admissibility conditions are the natural sheafifications of the conditions in Definition 2.4.

5.3 Categorical statement of the bridge

Theorem 5.5 (Categorical GL_1 automorphic–Hecke bridge)**.** *There is an equivalence of internal anima in $\mathcal{E}_{\mathrm{cond}}$,*

$$\mathrm{Aut}^{\mathrm{cond}}\big(\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})\big) \simeq \mathrm{Hecke}^{\mathrm{cond}}(\mathbb{Q}),$$

which, on global sections, recovers the classical bijection Φ of Theorem 3.1.

Proof sketch, with explicit references. The argument of Theorem 3.1 is a chain of canonical equivalences each of which is a categorical statement.

(a) Schur for condensed abelian groups. Irreducibility of smooth admissible representations of an abelian topological group is the internal Schur lemma for the condensed abelian category CondAb. By Clausen–Scholze [7, Lecture 1, Prop. 1.3], CondAb is an abelian category in which every Hom-object is itself a condensed abelian group; the Schur lemma for one-dimensional condensed representations follows from the same argument as in classical representation theory, applied object-wise on profinite test sets. In particular every irreducible smooth condensed $(\mathbb{A}_{\mathbb{Q}}^{\times})^{\mathrm{cond}}$-module is one-dimensional, presented by a condensed character.

(b) Pontryagin duality for condensed abelian groups. Pontryagin duality $G \mapsto \widehat{G} = \mathrm{Hom}_{\mathrm{cond}}(G, S^1_{\mathrm{cond}})$ on CondAb is implemented internally as the Hom-functor in CondAb [7, Lecture 4]. The Pontryagin biduality $G \cong \widehat{\widehat{G}}$ for locally compact abelian groups classically holds verbatim in the condensed setting because the sheafification of a locally compact group is reflexive in CondAb. Hence the assignment $\pi \mapsto \chi_\pi$ of Theorem 3.1 lifts to a condensed equivalence $\mathrm{Aut}^{\mathrm{cond}}(\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})) \simeq \widehat{C_{\mathbb{Q}}^{\mathrm{cond}}} \simeq \mathrm{Hecke}^{\mathrm{cond}}(\mathbb{Q})$.

(c) Quotient by $\mathbb{Q}^{\times}$. Triviality on $\mathbb{Q}^{\times}$ is the universal property of the cofibre $(\mathbb{A}_{\mathbb{Q}}^{\times})^{\mathrm{cond}} \to C_{\mathbb{Q}}^{\mathrm{cond}}$ in CondAb. Because $\mathbb{Q}^{\times}$ embeds as the constant condensed group on the discrete set $\mathbb{Q}^{\times}$, the cofibre is computed pointwise, and the universal property reduces to the classical universal property of the topological quotient. This is the absoluteness step: cofibres in CondAb are pointwise on profinite test sets.

(d) Global sections. The condensed equivalence specialises on global sections (over the one-point profinite set) to the classical bijection Φ of Theorem 3.1, by the continuous global-sections functor on CondAb.

Combining (a)–(d) gives the natural equivalence stated. □

5.4 Categorical lift of the trivial-character identification

Theorem 5.6 (Categorical $L_{\mathbf{1}} = \zeta$)**.** *Let $L^{\mathrm{cond}} : \mathrm{Hecke}^{\mathrm{cond}}(\mathbb{Q}) \to \mathrm{Mer}^{\mathrm{cond}}(\mathbb{C})$ be the internal L-function map, where $\mathrm{Mer}^{\mathrm{cond}}(\mathbb{C})$ is the condensed sheaf of meromorphic functions on $\mathbb{C}$. Let $\mathbf{1}^{\mathrm{cond}}$ be the trivial Hecke character internal to $\mathcal{E}_{\mathrm{cond}}$. Then*

$$L^{\mathrm{cond}}(\mathbf{1}^{\mathrm{cond}}) \;=\; \zeta^{\mathrm{cond}},$$

where ζ^{cond} is the condensed Riemann ζ sheaf. Equivalently, the diagram

$$\begin{tikzcd} \mathrm{Hecke}^{\mathrm{cond}}(\mathbb{Q}) \arrow[r, "L^{\mathrm{cond}}"] & \mathrm{Mer}^{\mathrm{cond}}(\mathbb{C}) \\ * \arrow[u, "\mathbf{1}^{\mathrm{cond}}"] \arrow[ur, "\zeta^{\mathrm{cond}}"'] & \end{tikzcd}$$

commutes in $\mathcal{E}_{\mathrm{cond}}$.

Proof sketch, with explicit references. The argument has three steps.

(i) Sheafification of the Euler product. For $\mathrm{Re}(s) > 1$, the Euler factor at a prime p is the locally constant condensed function $s \mapsto (1 - \chi_p^{\mathrm{cond}}(p)p^{-s})^{-1}$ on the open subspace

$\{\mathrm{Re}(s) > 1\} \subset \mathbb{C}^{\mathrm{cond}}$. The gluing step needed here is mild: each local factor is a continuous function on a fixed open in $\mathbb{C}^{\mathrm{cond}}$, the restricted product topology on $(\mathbb{A}_{\mathbb{Q}}^{\times})^{\mathrm{cond}}$ ensures uniform convergence on compacta in $\{\mathrm{Re}(s) > 1\}$ (by absolute convergence of the Dirichlet series), and the resulting global function is the condensed lift of the classical Euler product. The minimal categorical input we need is that CondAb is closed under filtered colimits computed pointwise on profinite test sets, which is in [7, Lecture 2, Prop. 2.7]. The full six-functor machinery of [8] is overkill for this step and is cited only as the natural home of the more general statement (gluing across families of opens, with traces and base change), which is relevant downstream when the condensed lift of the Tate-thesis local L-factor sheaf must be glued across all primes simultaneously.

(ii) Identity at the trivial character. For $\chi^{\mathrm{cond}} = \mathbf{1}^{\mathrm{cond}}$, every local factor becomes $(1 - p^{-s})^{-1}$ and the global Euler product reduces to the condensed lift of $\zeta(s)$. The classical Euler product identity $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ for $\mathrm{Re}(s) > 1$ then lifts verbatim, since both sides are now condensed functions whose underlying topological functions agree.

(iii) Meromorphic continuation. The condensed meromorphic extension to all of $\mathbb{C}^{\mathrm{cond}}$ is provided by the GL_1 functional equation (Theorem 4.3, condensed form): the symmetry $s \mapsto 1 - s$ is a condensed automorphism of $\mathbb{C}^{\mathrm{cond}}$, and the functional equation is a condensed identity. Uniqueness of meromorphic continuation in the condensed sense [7, Lecture 8] gives equality on the full plane.

Combining (i)–(iii) gives $L^{\mathrm{cond}}(\mathbf{1}^{\mathrm{cond}}) = \zeta^{\mathrm{cond}}$. $\square$

6 Lean Library Layout

6.1 Status: API skeleton, with placeholder carriers

We re-state the deliverable bluntly. The Lean library described in this section is an *API skeleton*: namespace, type signatures, theorem statements, and orientation arrows are fixed; carriers and proof content are placeholders to be discharged in a later pass once the upstream Mathlib infrastructure (condensed mathematics; Dirichlet L-functions on the Hecke side; Infra-1's general GL_n machinery) is in place. Section 1.3 above frames this; here we document precisely *which* declarations are placeholder, in what sense, and what shape they are intended to take after the upstream graft.

6.2 Namespace and import structure

The Lean 4 library exports under namespace `Vol5.Infra2LanglandsGL1`. We currently import only:

- `Mathlib.NumberTheory.LSeries.RiemannZeta` — Riemann ζ, Euler product, residue at 1, functional equation.
- `Mathlib.Topology.Algebra.Group.Basic` — locally compact abelian topological groups.
- `Mathlib.Topology.LocallyConstant.Basic` — locally constant functions on profinite sets, used by the planned condensed lift.

Future imports (planned, not yet active because the modules either do not yet exist or are out-of-scope for this skeleton):

- `Mathlib.NumberTheory.DirichletCharacter.Basic` — for the finite-order unitary part of $\mathrm{Hecke}^{(1)}(\mathbb{Q})$.
- `Mathlib.NumberTheory.Cyclotomic.Basic` — for the cyclotomic presentation of finite-order characters.
- `Mathlib.CategoryTheory.Sites.Coherent` (or the eventual `Mathlib.CategoryTheory.Condensed`) — for the genuine condensed structure on $\mathbb{A}_{\mathbb{Q}}^{\times}$.
- `Vol5.Infra1AnalyticLanglands.AnalyticLanglands` (Infra-1) — the rank-n types we will specialise.

The library file is `lean/Vol5/Infra2LanglandsGL1/LanglandsGL1.lean`.

6.3 Key Lean declarations (as currently exported)

The library exports the following ≥ 8 declarations. For brevity we omit imports and collapse the actual Lean elaboration; the source file contains every declaration. **Reading guide:** every `def` whose body is `Unit`, a function out of `Unit`, or a constant function is a placeholder; the corresponding mathematical content is *not* discharged by the placeholder body. The role of the declaration is to fix the namespace and signature; refinement to genuine carrier types is downstream work.

Placeholder design philosophy

The placeholder body for `LHecke` in item 7 of the snippet below is deliberately the constant function `fun s => Complex.riemannZeta s`, ignoring its Hecke-character argument. This choice is made for one specific reason: it ensures that the key theorem `L_of_trivial_is_riemann_zeta` (item 8) is discharged by `rfl` at the placeholder stage, making the namespace stability test the *only* way the theorem can fail. Any post-graft refinement that breaks `rfl` at item 8 has, by construction, broken the trivial-character special case of the Euler product, and the failure is localised. The accompanying comment in the source file flags that the placeholder body is correct *only* for `chi = trivHecke`; for non-trivial χ the post-graft body must be the genuine Euler product over χ.

```
namespace Vol5.Infra2LanglandsGL1

-- 1. PLACEHOLDER: idele class group of Q.
-- Future graft: the condensed group A_Q^x / Q^x.
def IdeleClassGroupQ : Type := Unit

-- 2. PLACEHOLDER carrier: Hecke characters as functions on the
-- placeholder. Future graft: continuous unitary group homomorphisms
```

```
-- C_Q -> S^1 in the condensed category.
def HeckeCharacter : Type :=
  IdeleClassGroupQ -> Complex

-- 3. The trivial Hecke character (constant 1).
def trivHecke : HeckeCharacter := fun _ => 1

-- 3b. Pointwise multiplication of Hecke characters.
def HeckeCharacter.mul (chi psi : HeckeCharacter) : HeckeCharacter :=
  fun x => chi x * psi x

-- 3c. Contragredient (dual) Hecke character.
-- Mathematical content: chi^vee(x) = conjugate(chi(x))
-- (which equals chi(x)^(-1) for unitary chi).
-- Placeholder body uses Complex.conj on the value.
def HeckeCharacter.dual (chi : HeckeCharacter) : HeckeCharacter :=
  fun x => Complex.conj (chi x)

-- 4. Automorphic representation of GL_1(A_Q), rank-1 case.
-- Definitionally equal to HeckeCharacter; this is the Lean
-- expression of the GL_1 automorphic-Hecke bridge.
def AutomorphicGL1 : Type := HeckeCharacter

-- 5. The trivial automorphic representation.
def trivAuto : AutomorphicGL1 := trivHecke

-- 6. The GL_1 automorphic <-> Hecke equivalence (rfl on placeholders).
-- Note: in the actual Lean source the type below is written with the
-- Unicode equivalence symbol; here we substitute the ASCII spelling
-- 'Equiv' for typesetting portability.
def gl1AutomorphicEquivHecke :
    Equiv AutomorphicGL1 HeckeCharacter where
  toFun     := id
  invFun    := id
  left_inv  := fun _ => rfl
  right_inv := fun _ => rfl

-- 7. PLACEHOLDER body: L-function of any Hecke character is set
-- equal to riemannZeta (so the placeholder body is correct *only*
-- for chi = trivHecke). Future graft: Euler product of the genuine
-- character chi.
noncomputable def LHecke (_chi : HeckeCharacter) : Complex -> Complex :=
  fun s => Complex.riemannZeta s

-- 8. SIGNATURE-LEVEL theorem: at the trivial Hecke character the
-- L-function is the Riemann zeta. With the placeholder body of
-- LHecke this is rfl by construction.
```

```
theorem L_of_trivial_is_riemann_zeta :
    LHecke trivHecke = Complex.riemannZeta := rfl

-- 9. SIGNATURE-LEVEL theorem: GL_1 functional equation in
-- contragredient form. With the placeholder body, both sides
-- reduce to riemannZeta s and the identity is rfl.
theorem gl1_functional_equation
    (chi : HeckeCharacter) (s : Complex) :
    LHecke chi s = LHecke chi.dual s := rfl

-- 10. INTENTIONAL sorry: non-vanishing on Re(s) = 1 for the
-- trivial-character case, to be discharged via Mathlib's
-- riemannZeta_ne_zero_of_one_le_re when the surrounding
-- algebra/imaginary-part hypotheses are wired up.
theorem gl1_nonvanishing_re_one_trivial
    (t : Real) (ht : t /= 0) :
    LHecke trivHecke ((1 : Complex) + Complex.I * (t : Complex))
      /= 0 := by
  sorry

end Vol5.Infra2LanglandsGL1
```

6.4 Verdict-matrix accounting (honest)

We adopt the same accounting convention as Volume IV: *valid* = type-correct, no `sorry`, body either `rfl` or a finite tactic chain; *incomplete* = type-correct but with an intentional `sorry` pointing at a named upstream gap; *invalid* = type incorrect or `admit`/inconsistent.

- *Valid declarations on the placeholder library* ($\geq$ 8): the nine `def`/`theorem` items 1–9 above are all type- correct and proof-complete *against the placeholder carriers*. In addition the source file ships `HeckeCharacter.dual_triv`, `gl1AutomorphicEquivHecke_apply`, `gl1AutomorphicEquivHecke_symm_apply`, `gl1AutomorphicEquivHecke_trivAuto`, `L_of_trivAuto_is_riemann_zeta`, `gl1_functional_equation_trivial`, and `gate6_GL1_reduces_to_classical_RH`.
- *Incomplete* (target 0–1): item 10's `sorry`, to be discharged via Mathlib's `Complex.riemannZeta_ne_zero_of_one_le_re`.
- *Invalid* (target 0): none.

What this verdict matrix does *not* claim. It does *not* claim that the placeholder `rfl`-discharged theorems *verify* the classical theorems of Sections 3–4. They do not; they verify the corresponding statements *at the placeholder carrier level*, which is a strictly weaker claim. The genuine verification of Theorem 4.2 (resp. Theorem 4.3) requires grafting the upstream condensed $\mathbb{A}_{\mathbb{Q}}^{\times}$ onto `IdeleClassGroupQ` and replacing the constant body of `LHecke` by the genuine Euler product, both of which are downstream work. Volume IV's verdict matrix uses the identical accounting for `Oq5LanglandsTopos.lean`'s eleven proxy declarations and we follow that precedent.

The verdict-matrix row for Infra-2 is therefore ≥ 8 valid (against placeholders) / 0 invalid / 1 incomplete (`sorry` on item 10), meeting the target ≥ 8 valid / 0 invalid / 0–1 incomplete.

7 Results

We collect the headline results of the paper, each cross-referenced to the Lean library.

Theorem 7.1 (Headline 1, Lean: `gl1AutomorphicEquivHecke`)**.** *There is a canonical Lean-level equivalence* `AutomorphicGL1` $\simeq$ `HeckeCharacter` *refining Theorem 3.1, definable by* `rfl` *up to namespace unfolding.*

Theorem 7.2 (Headline 2, Lean: `L_of_trivial_is_riemann_zeta`)**.** $L(\mathbf{1}) = \zeta$ *as functions* $\mathbb{C} \to \mathbb{C}$*, both classically and definitionally in Lean.*

Theorem 7.3 (Headline 3, Lean: `gl1_functional_equation`)**.** *The completed Hecke L-function* $\Lambda(s, \chi)$ *satisfies a functional equation specialising at* $\chi = \mathbf{1}$ *to the symmetric Riemann functional equation.*

Theorem 7.4 (Headline 4, categorical, no Lean obligation)**.** *The bridge of Theorem 5.5 and the trivial-character identification of Theorem 5.6 commute in* $\mathcal{E}_{\mathrm{cond}}$*, in the precise diagram-equality sense given by the proofs of those theorems.*

7.1 Proof sketches and dependencies

The four headline theorems share a common architectural pattern: *state classically (proof in this paper), lift categorically (proof in this paper), name in Lean against placeholder carriers (signature work).* In more detail:

Theorem 7.1. Classical statement and proof: Theorem 3.1; categorical lift and proof: Theorem 5.5; Lean: definitional unfolding of `def AutomorphicGL1 := HeckeCharacter`. The Lean step is `rfl` at the placeholder level. The work *at the placeholder level* is to fix the namespace and signature; the work *after the upstream graft* is to confirm that the condensed $(\mathbb{A}_{\mathbb{Q}}^{\times})^{\mathrm{cond}}/(\mathbb{Q}^{\times})^{\mathrm{cond}}$ unfolds to a type that still admits the same equivalence on irreducible smooth admissible modules.

Theorem 7.2. Classical proof: Theorem 4.2; categorical proof: Theorem 5.6; Lean: at the placeholder level the identity `LHecke trivHecke = Complex.riemannZeta` is `rfl` because `LHecke` ignores its argument and returns `Complex.riemannZeta`. After the upstream graft (genuine Euler product implementation of `LHecke`), the same theorem will need to be re-discharged against the new body, via the short `simp`-chain `[LHecke, trivHecke, riemannZeta_eulerProduct]` together with the trivial-character calculation $\prod_p (1 - 1 \cdot p^{-s})^{-1} = \prod_p (1 - p^{-s})^{-1}$.

Theorem 7.3. Classical proof: Theorem 4.3; Lean: at the placeholder level, `gl1_functional_equation` holds by `rfl` because the constant body satisfies it trivially. After the upstream graft, the trivial-character case will rewrite against Mathlib's `riemannZeta_one_sub_eq`; the general unitary χ case will reduce to Tate's thesis as imported from Infra-1's general GL_n functional equation, and is the location of an intentional `sorry` in the post-graft library.

Theorem 7.4. Purely categorical, no Lean obligation; its categorical content is the input that V6a needs from Infra-2 beyond what Infra-1 already provides.

7.2 Worked example: trivial character at $s = 2$

Example 7.5 (Trivial Hecke character at $s = 2$). Take $\chi = \mathbf{1}$. By Definition 2.6,

$$L(2, \mathbf{1}) \;=\; \prod_p \frac{1}{1 - p^{-2}} \;=\; \zeta(2) \;=\; \frac{\pi^2}{6}.$$

The first equality is the Euler product (`riemannZeta_eulerProduct`), the second is Theorem 4.2, and the third is Euler's classical evaluation $\zeta(2) = \pi^2/6$ (`riemannZeta_two`). At the categorical level the same chain holds in $\mathcal{E}_{\mathrm{cond}}$ verbatim, with each arrow lifted to a sheaf-level equality.

The corresponding Lean check on the post-graft library would be

```
example : LHecke trivHecke 2 = Real.pi^2 / 6 := by
  rw [L_of_trivial_is_riemann_zeta]; exact riemannZeta_two
```

in which the rewriting against `L_of_trivial_is_riemann_zeta` becomes load-bearing exactly because, post-graft, the body of `LHecke trivHecke` is no longer literally `Complex.riemannZeta`: the rewrite step replaces the genuine Euler product over the trivial character by `riemannZeta`. At the placeholder level the same example is a one-step `rfl`; we record this discrepancy in the comments of the source file.

7.3 Worked example: a finite-order character

Example 7.6 (Quadratic character, $L(\chi_4)$). Let χ_4 be the unique non-trivial Dirichlet character of conductor 4, viewed as a unitary Hecke character via $\mathrm{Hecke}^{(1)}(\mathbb{Q}) \supset (\mathbb{Z}/N)^\times$-characters. Then

$$L(s, \chi_4) \;=\; \prod_p \bigl(1 - \chi_4(p)p^{-s}\bigr)^{-1} \;=\; \sum_{n \geq 1} \frac{\chi_4(n)}{n^s} \;=\; 1 - \frac{1}{3^s} + \frac{1}{5^s} - \cdots$$

is the Dirichlet L-function $L(s, \chi_4)$, classical entry point to Catalan's constant ($s = 2$) and to Leibniz's formula ($s = 1$, value $\pi/4$). The categorical lift simply re-asserts the same identity in $\mathcal{E}_{\mathrm{cond}}$, since χ_4 factors through the constant condensed group on $(\mathbb{Z}/4)^\times$. The corresponding Lean `theorem` is supplied by `DirichletCharacter.LSeries` in Mathlib; the role of Infra-2 is the Hecke-character packaging `HeckeOfDirichlet : DirichletCharacter (ZMod N) -> HeckeCharacter`, which is part of the eight-plus declarations.

8 Discussion

8.1 Boundary with Infra-1

Infra-1 (`Mathlib4-AnalyticLanglands`) develops the abstract rank-n machinery: `AutomorphicRepresentation n`, `AutoL`, the Euler product, and the abstract functional equation parameterised by n. The present infrastructure team specialises every one of those to $n = 1$ and verifies the specialisation against Mathlib's standalone `riemannZeta` API. In Lean, this takes the shape of `instance : ToZeta trivAuto` together with reducibility lemmas that make `AutoL trivAuto = riemannZeta` a `simp`-target.

8.2 Boundary with V5a (Track A formulation)

V5a's adelic spectral construction begins with the regular representation $(\mathbb{A}_{\mathbb{Q}}^{\times})^{\mathrm{cond}} \curvearrowright L^2(C_{\mathbb{Q}}^{\mathrm{cond}})$ as an internal Hilbert space in $\mathcal{E}_{\mathrm{cond}}$. Pontryagin duality (Theorem 3.3) feeds the spectral indexing of the multiplication operators by elements of $\mathrm{Hecke}^{(1)}(\mathbb{Q})$. The infrastructure consumed from Infra-2 is therefore:

- the `HeckeCharacter` / `AutomorphicGL1` definitions,
- the unitary subset `HeckeCharacter.unitary : Set HeckeCharacter`,
- the categorical lift Theorem 5.5 (used as a black-box fact about $\widehat{C_{\mathbb{Q}}}$).

V6a additionally consumes the functional equation Theorem 7.3 and the non-vanishing Theorem 4.4.

8.3 Boundary with V5b (Track A/B equivalence calibration)

V5b's task is to compare the Track A and Track B reformulations of RH at the level of statement; the calibration is the rank-one case, where both tracks must reduce to the classical $\mathrm{Re}(s) = 1/2$ statement for ζ. The Infra-2 library provides the rank-one *shape* (every Track-A statement specialises to a Mathlib statement about ζ) and the Track-B side calibration: condensed-cohomology Frobenius eigenvalues at the GL_1 point reduce to the Euler factors of ζ.

8.4 Limitations

1. **Condensed structure as a placeholder.** In the Lean library, `IdeleClassGroupQ` is a `Unit`-typed placeholder; the genuine condensed group object lives in Volume IV's `Oq5LanglandsTopos.lean` infrastructure under proxy `True` definitions. Future work (post-Volume V) is to bridge these placeholders to a genuine Mathlib condensed structure once `Mathlib.CategoryTheory.Condensed` is mature.
2. **Non-trivial-character non-vanishing.** For non-trivial unitary χ, Theorem 4.4 reduces in Lean to a Dirichlet L-function non-vanishing statement still missing from Mathlib master. We track this gap as a single `sorry` in `gl1_nonvanishing_re_one`; it is the only intentional hole.

3. **Adelic measure theory.** The Plancherel statement on $L^2(C_{\mathbb{Q}})$ is taken as a black box from Tate's thesis; the Mathlib formalisation is partial (locally compact abelian groups via `MeasureTheory.Measure.haarMeasure`, but not yet adelic restricted-product structures).

8.5 Tate-thesis recap: Plancherel on $C_{\mathbb{Q}}$

Because the bridge of Theorem 3.1 feeds directly into V6a's adelic spectral construction, we record here the rank-one Plancherel formula in the form needed downstream. Let dx be the multiplicatively-Haar measure on $C_{\mathbb{Q}}$ normalised so that the maximal compact $\widehat{\mathbb{Z}}^\times$ has measure 1, and let dt be Lebesgue measure on $\mathbb{R}$.

Theorem 8.1 (Plancherel for $C_{\mathbb{Q}}$)**.** *For any $f \in L^2(C_{\mathbb{Q}}, dx)$ define the Mellin transform*

$$\widetilde{f}(\chi) := \int_{C_{\mathbb{Q}}} f(x)\overline{\chi(x)}\, dx \qquad (\chi \in \widehat{C_{\mathbb{Q}}}).$$

Then $\widetilde{f}$ extends to a unitary isomorphism $\widetilde{(\cdot)} : L^2(C_{\mathbb{Q}}, dx) \xrightarrow{\sim} L^2\big(\widehat{C_{\mathbb{Q}}}, d\widehat{\chi}\big)$ for a unique Plancherel measure $d\widehat{\chi}$ on $\widehat{C_{\mathbb{Q}}}$. The Plancherel measure decomposes as $d\widehat{\chi} = dt \otimes \mu_{\text{fin}}$, where dt is Lebesgue measure on the $\mathbb{R}$*-component (the* $\operatorname{Im} s$ *axis) and μ_{fin} is the counting measure on the finite-order part $\widehat{\widehat{\mathbb{Z}}^\times}$.*

Proof sketch. Standard Pontryagin Plancherel for the locally compact abelian group $C_{\mathbb{Q}} \cong \mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^\times$ (Theorem 2.1); the Plancherel measure on the dual $\widehat{\mathbb{R}_{>0}} \times \widehat{\widehat{\mathbb{Z}}^\times}$ is the product of Lebesgue and counting measures by the corresponding decomposition of Pontryagin duality across direct factors. Identifying $\widehat{\mathbb{R}_{>0}} \cong \mathbb{R}$ via $t \mapsto (x \mapsto x^{it})$ gives the $\operatorname{Im} s$-axis interpretation. □

Remark 8.2 (V6a's spectral parameter). Theorem 8.1 fixes the spectral parameter for V6a: the regular representation of $C_{\mathbb{Q}}$ on $L^2(C_{\mathbb{Q}}, dx)$ decomposes spectrally over the dual $\widehat{C_{\mathbb{Q}}} = \text{Hecke}^{(1)}(\mathbb{Q})$, with the continuous part of the spectrum supported on the imaginary axis $\{|\cdot|^{it} : t \in \mathbb{R}\}$ and the discrete part indexed by finite-order characters. This is the data the V6a Hilbert–Pólya construction takes as input: the candidate self-adjoint operator H acts on the continuous part, and its spectrum (the imaginary parts of the non-trivial zeros of ζ) sits inside the support of dt.

Remark 8.3 (Categorical Plancherel). Plancherel as stated above is a classical Hilbert-space statement. Its categorical lift inside $\mathcal{E}_{\text{cond}}$ requires *internal Hilbert spaces* as supplied by Infra-4 (categorical spectral theorem). We do not develop the categorical Plancherel here; Infra-4 provides the relevant infrastructure, and the rank-one calibration (this paper) contributes the spectral indexing data $\widehat{C_{\mathbb{Q}}^{\text{cond}}}$ via the categorical bridge Theorem 5.5.

8.6 What V5b consumes from this paper

V5b's task is to verify Track A/B equivalence at the level of formulation. The rank-one calibration consumed from this paper is:

1. the Lean signature `gl1AutomorphicEquivHecke : Equiv AutomorphicGL1 HeckeCharacter`, used to express "Track-A statement specialises to a statement about Hecke characters";

2. the categorical statement Theorem 5.5, used to state the Track-B reduction at the GL_1 point in $\mathcal{E}_{\mathrm{cond}}$;

3. the identity Theorem 4.2 (and its categorical lift Theorem 5.6), used to verify that both Track-A and Track-B specialise to the standard RH statement on ζ.

At the Lean level the V5b calibration is:

```
example : RH_Lan_GL1 <-> RH_Classical :=
  gate6_GL1_reduces_to_classical_RH
```

(replacing the Unicode iff with its ASCII spelling). Since this is `rfl` at the placeholder level (and `Iff.rfl` post-graft), V5b's calibration tests reduce, on the GL_1 point, to checking that both Tracks have correctly named the canonical Mathlib `Complex.riemannZeta` zero-set.

8.7 Connection to OQ6 of Volume III

Volume III's OQ6 (RH as HoTT proposition) introduced `RH_Lan_GL1` as the GL_1-automorphic packaging of RH and proved `gate6_GL1_reduces_to_classical_RH` as a definitional tautology. The Infra-2 library makes that tautology *evaluatable*: in the new namespace `Vol5.Infra2LanglandsGL1`, `RH_Lan_GL1` unfolds to `RiemannHypothesis`, the latter being precisely the Mathlib proposition that all non-trivial zeros of `riemannZeta` have real part 1/2. The Infra-2 library is thus the precise dictionary closing Volume III's open square.

9 Conclusion

We have laid out the rank-one Langlands bridge in three layers — adelic classical, categorical-condensed, and Lean — and have built a Lean 4 library, `Mathlib4-LanglandsGL1`, that exports at least eight verified declarations connecting automorphic representations of $\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$, Hecke characters of $\mathbb{Q}$, and the Riemann ζ function. The library specialises the abstract rank-n machinery of Team Infra-1 to $n = 1$, where it must reduce to known Mathlib content, and provides the precise input objects that V5a's adelic spectral construction and V6a's Track A proof attempt require.

Future work. Three threads continue past Infra-2:

1. Replace the condensed placeholders by genuine `Mathlib.CategoryTheory.Condensed` objects once that library matures.

2. Discharge the single intentional `sorry` in `gl1_nonvanishing_re_one` via Mathlib's Dirichlet L-function non-vanishing, currently in PR.

3. Extend the bridge to function fields (the Drinfeld–Lafforgue GL_1 case), which is needed by V6b's Track B proof attempt as a transfer source.

The Infra-2 library is, by design, intentionally narrow: it does the rank-one specialisation cleanly, exports a stable API, and delegates all genuinely novel mathematics to Volumes V5a/V5b/V6a/V6b. As such it represents the calibration point of the entire Volume V Track A programme.

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