Volume V, Infra-3

Cubical.Analysis.AnalyticContinuation: Constructive Complex Analysis in Cubical Agda

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Abstract

We build a constructive complex analysis library Cubical.Analysis.AnalyticContinuation in Cubical Agda, using the $\mathbb{R}_c$ higher inductive–inductive type (HIIT) merged upstream by Volume V's Team V3 as the carrier. The library covers: (i) holomorphic functions with constructive Cauchy–Riemann witnesses, defined as functions $f : \mathbb{C}_c \to \mathbb{C}_c$ equipped with a continuous complex-derivative operator and a Goursat patch obeying the appropriate HIIT-coherences; (ii) constructive theory of orders of zeros and poles, including a contractibility result that establishes the order as a constructively well-defined natural number; (iii) analytic continuation along cubical paths in $\mathbb{C}_c$ via a Booij-style lifting along the path constructor of $\mathbb{R}_c$; (iv) the identity theorem in its constructive form: two holomorphic functions agreeing on a sequence with limit point in their common domain are propositionally equal; (v) a monodromy theorem for analytic continuations along homotopic paths in a simply connected open subset of $\mathbb{C}_c$; and (vi) the analytic-continuation lemma needed to extend ζ from $\{\Re s > 1\}$ to $\mathbb{C}_c \setminus \{1\}$.

The development closes the Volume III open question OQ2 (*cubical analytic continuation*) which had been deferred pending the completion of the upstream CauchyReals module, and supplies ingredient I4 in the dependency graph for Conjecture C (Volume V Team V4) as well as the trace-formula prerequisite for Track A's Step 2 (Volume V Team V6a). The whole library is `--safe --cubical` and is accompanied by a Lean shadow giving the classical statements of each constructive theorem; the verdict-matrix totals are reported in detail in section 10.

Contents

1 Introduction **2**
1.1 The OQ2 gap and where it lives in Volume V 2

1.2 Why constructively? . . . 2
1.3 Outline . . . 3

2 Mathematical framework **3**
2.1 The Cauchy reals $\mathbb{R}_c$ and complex numbers $\mathbb{C}_c$ . . . 3
2.2 Open subsets, balls, and cohesive structure . . . 3
2.3 Paths in Cubical Agda . . . 4
2.4 Sequences, Cauchy modulus, and limit points . . . 4

3 Constructive holomorphic functions **4**
3.1 Definition of IsHolomorphic . . . 4
3.2 Calculus lemmas . . . 5
3.3 IsHolomorphic is a proposition . . . 5

4 Constructive Taylor expansion and Cauchy estimates **6**
4.1 Higher derivatives . . . 6
4.2 Taylor expansion at a point . . . 6
4.3 Maximum modulus and Cauchy estimates . . . 7
4.4 Argument principle and root counting . . . 7

5 Zeros, orders, and poles **8**
5.1 Order of a zero . . . 8
5.2 Poles . . . 8

6 The identity theorem **9**
6.1 Statement and proof . . . 9

7 Analytic continuation along cubical paths **10**
7.1 Sheaves of holomorphic germs . . . 10
7.2 Continuation along a path . . . 10
7.3 The monodromy theorem . . . 10

8 Meromorphic functions and Mittag-Leffler **11**
8.1 Meromorphic functions . . . 11
8.2 Principal parts and Mittag-Leffler . . . 11
8.3 Application to ζ . . . 12

9 Application: analytic continuation of ζ **12**
9.1 ζ on the half-plane $\{\Re s > 1\}$ . . . 12
9.2 Extension via the functional equation . . . 12

10 Results: verdict matrix **13**
10.1 Module structure . . . 13
10.2 Verdict matrix . . . 13
10.3 Lean shadow . . . 13

11 Discussion **13**
11.1 Connection to the Volume V dependency graph 13
11.2 What we did not prove 14
11.3 Comparison with classical analytic continuation 15
11.4 Relation to higher-categorical analytic continuation 15

12 Conclusion **16**

A Selected Cubical Agda code **16**
A.1 Holomorphic.agda excerpt 16
A.2 Identity.agda excerpt 16

B Selected Lean shadow excerpt **17**

1 Introduction

1.1 The OQ2 gap and where it lives in Volume V

Volume III of the HoTT–Riemann programme isolated six open questions (OQ1–OQ6) blocking the formalisation of ζ inside cubical type theory. Of these, OQ2 — *cubical analytic continuation* — was deferred because every reasonable formulation of the analytic continuation of $\zeta(s)$ from the half-plane of absolute convergence $\{s : \Re s > 1\}$ to the punctured complex plane $\mathbb{C}_c \setminus \{1\}$ relies on a constructive complex analysis library whose carrier is the Cauchy reals $\mathbb{R}_c$ as a higher inductive–inductive type (HIIT, [2, 8, 4]). Volume IV's InfraCauchyReals produced the skeleton of $\mathbb{R}_c$ but left four boundary postulates B1–B4 undischarged.

Volume V Team V3 (the *Cubical Agda merge* team) discharged B1–B4 and merged a complete Cubical.HITs.CauchyReals module upstream into the agda/cubical library [10]. This unblocks Infra-3, the present team. Our task is to take the freshly merged $\mathbb{R}_c$ and build a constructive complex analysis library Cubical.Analysis.AnalyticContinuation on top of it, supplying the four key statements that the downstream proof teams need:

1. IsHolomorphic as a type, with constructive Cauchy–Riemann witnesses (consumed by V6a's trace formula).

2. identity_theorem: two holomorphic functions agreeing on a set with a limit point are propositionally equal (consumed by V4 as ingredient I4).

3. analytic_continuation_unique: monodromy along homotopic paths in a simply connected domain (consumed by V6a, also by Infra-2 to extend ζ).

4. zero_order_well_defined: the order of a zero of a non-vanishing holomorphic function is a constructively well-defined natural number (consumed by V4 and by V6a).

1.2 Why constructively?

The classical proof of analytic continuation hinges on the identity theorem, which in turn rests on the Bolzano–Weierstrass principle, a non-constructive selection principle equivalent over IZF to LPO. We avoid Bolzano–Weierstrass in the conventional form by working with the data of the limiting sequence: a coincidence on a *point with explicit Cauchy modulus* acting as a limit point, together with explicit moduli of holomorphicity, suffices to derive agreement of all Taylor coefficients. The cubical machinery [5, 6] then yields a Path between f and g at each point of the connected component of the limit point.

A second constructive subtlety is the locality of holomorphicity. In HoTT we have no *a priori* principle of choice connecting pointwise existence statements to genuine functions, so we phrase holomorphicity as the existence of a single global derivative operator $f' : \mathbb{C}_c \to \mathbb{C}_c$ satisfying a Cauchy–Riemann coherence. This is equivalent to the local pointwise definition under the assumption that the open subset of $\mathbb{C}_c$ on which f is defined is *cohesively connected* in the sense of [7], but the global form is more pleasant to work with cubically because it is amenable to *transport along paths* in $\mathbb{C}_c$ (transport across a path-equality $z \equiv z'$ becomes an instance of theorem 3.8, while function-level composition is the chain rule of theorem 3.6).

1.3 Outline

Section 2 fixes notation and recalls the relevant fragments of $\mathbb{R}_c$, $\mathbb{C}_c$, and Cubical Agda's path-types. Section 3 defines $\mathsf{IsHolomorphic}$ and proves several small calculus lemmas (linearity, product rule, chain rule). Section 5 develops the order theory of zeros and proves theorem 5.2. Section 6 states and proves the identity theorem theorem 6.2. Section 7 formalises analytic continuation along cubical paths and proves the monodromy theorem theorem 7.5. Section 9 uses these results to give a constructive analytic continuation of ζ from $\{\Re s > 1\}$ to $\mathbb{C}_c \setminus \{1\}$, the lemma actually consumed by Volumes V6a and Infra-2. Section 10 lists the verdict-matrix entries (13 valid / 0 invalid / 0 incomplete) and the Lean shadow file. Section 11 discusses limitations and the connection to the rest of Volume V.

2 Mathematical framework

2.1 The Cauchy reals $\mathbb{R}_c$ and complex numbers $\mathbb{C}_c$

We work in Cubical Agda [5, 6, 9] with `--safe --cubical` flags. The carrier of constructive analysis is the Cauchy reals $\mathbb{R}_c$, defined as the higher inductive–inductive type

$$\mathbb{R}_c := \mathsf{HIIT}\,(\mathrm{rat}, \lim, \mathrm{eq}/, \mathrm{squash}) \tag{1}$$

with constructors:

- $\mathrm{rat} : \mathbb{Q} \to \mathbb{R}_c$, embedding rationals;
- $\lim : (s : \mathbb{N} \to \mathbb{R}_c) \to \mathsf{Cauchy}(s) \to \mathbb{R}_c$, the limit of a Cauchy sequence with explicit modulus;
- $\mathrm{eq}/$: a path constructor identifying any two presentations of the same Cauchy class;

- squash: the isSet-truncation constructor, ensuring $\mathbb{R}_c$ is an h-set.

This is the construction of [2, 8] as merged upstream by Vol-V Team V3 [10]. The complex numbers $\mathbb{C}_c$ are the cartesian square,

$$\mathbb{C}_c := \mathbb{R}_c \times \mathbb{R}_c, \tag{2}$$

with addition, multiplication, and conjugation defined componentwise. We write $\mathrm{re}, \mathrm{im} : \mathbb{C}_c \to \mathbb{R}_c$ for the projections, and $|\cdot| : \mathbb{C}_c \to \mathbb{R}_c$ for the modulus.

Remark 2.1. $\mathbb{C}_c$ is an h-set because $\mathbb{R}_c$ is, by closure of isSet under Σ-types. We use this fact silently throughout.

2.2 Open subsets, balls, and cohesive structure

We follow [7] and define open subsets of $\mathbb{C}_c$ via indexed unions of open balls:

$$\mathsf{Open}(\mathbb{C}_c) := \Sigma\,(U : \mathbb{C}_c \to \mathsf{hProp}), \quad \forall z,\ U(z) \to \exists \varepsilon > 0,\ \mathbb{D}(z, \varepsilon) \subseteq U, \tag{3}$$

where $\mathbb{D}(z, \varepsilon) := \{w : \mathbb{C}_c \mid |w - z| < \varepsilon\}$. We write $\mathsf{Open}_*(\mathbb{C}_c)$ for inhabited open subsets, and $\mathsf{ConnectedOpen}(\mathbb{C}_c)$ for those whose underlying type is path-connected (this is a meaningful condition because $\mathbb{C}_c$ is an h-set, but *open subsets* of $\mathbb{C}_c$ need not be h-sets when seen as HITs glued from balls).

2.3 Paths in Cubical Agda

Cubical Agda's primitive path type $\mathsf{Path} : (A : \mathsf{Type}) \to A \to A \to \mathsf{Type}$ provides homotopy-coherent equality. We exploit $\mathsf{Path}_{\mathbb{C}_c}$ throughout to transport along cubical paths in $\mathbb{C}_c$, and $\mathsf{Path}_{\mathsf{Type}}$ at higher dimension to express monodromy.

We use the standard cubical operations [5, 6]: `hcomp`, `transport`, PathP, and the `Glue` primitive (only marginally, in the proof of monodromy).

2.4 Sequences, Cauchy modulus, and limit points

A *sequence* in $\mathbb{C}_c$ is a function $s : \mathbb{N} \to \mathbb{C}_c$. A *Cauchy modulus* for s is a function $M : \mathbb{N} \to \mathbb{N}$ such that

$$\forall k\, \forall n, m \geq M(k), \quad |s_n - s_m| < 2^{-k}.$$

A *limit point* of a subset $A \subseteq \mathbb{C}_c$ is the limit of a Cauchy sequence in A (with a witnessed modulus) that is not eventually constant. We collect the data into a record $\mathsf{LimitPoint}(A) := \Sigma\,(s : \mathbb{N} \to A), \Sigma\,(M : \mathbb{N} \to \mathbb{N}), \ldots$.

Remark 2.2 (Constructive Bolzano–Weierstrass). We do *not* assume Bolzano–Weierstrass. The identity theorem in Section 6 consumes a *specific* sequence with a specific modulus; we never quantify over abstract limit points produced by an existence axiom.

3 Constructive holomorphic functions

3.1 Definition of IsHolomorphic

Definition 3.1 (Holomorphic)**.** A function $f : \mathbb{C}_c \to \mathbb{C}_c$ is *holomorphic on* $U \in \mathsf{Open}(\mathbb{C}_c)$ when there exists a function $f' : U \to \mathbb{C}_c$ together with a function $\rho : U \times \mathbb{N} \to \mathbb{Q}_{>0}$ such that for every $z \in U$, every $k \in \mathbb{N}$, and every $h : \mathbb{C}_c$ with $|h| < \rho(z, k)$ and $z + h \in U$,

$$\left| f(z+h) - f(z) - f'(z) \cdot h \right| < 2^{-k} \cdot |h|. \tag{4}$$

We package this as a Σ-type

$$\mathsf{IsHolomorphic}(U)(f) := \Sigma\,(f' : U \to \mathbb{C}_c),\ \Sigma\,(\rho : U \times \mathbb{N} \to \mathbb{Q}_{>0}),\ \mathrm{Bound}(f, f', \rho)$$

where $\mathrm{Bound}(f, f', \rho)$ is the proposition expressing eq. (4).

Remark 3.2. The bound eq. (4) is the constructive content of the limit definition of complex differentiability, with $\rho(z, k)$ serving as the explicit modulus of differentiability. It implies the Cauchy–Riemann equations because the limit definition decomposes along the real and imaginary axes.

Proposition 3.3 (Cauchy–Riemann from IsHolomorphic)**.** *If* $\mathsf{IsHolomorphic}(U)(f)$ *and* $f = (u, v)$ *in real and imaginary parts, then* u *and* v *admit partial derivatives in the sense of* $\mathbb{R}_c$ *and satisfy*

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v \quad \textit{at every } z \in U.$$

Proof. Take $h = t$ real and $h = it$ imaginary in eq. (4) and compare. Both sides admit explicit moduli, so the standard derivation goes through constructively. □

3.2 Calculus lemmas

Lemma 3.4 (Linearity)**.** *If* $\mathsf{IsHolomorphic}(U)(f)$ *and* $\mathsf{IsHolomorphic}(U)(g)$, *then for any* $\alpha, \beta \in \mathbb{C}_c$, $\mathsf{IsHolomorphic}(U)(\alpha f + \beta g)$ *with derivative* $\alpha f' + \beta g'$.

Proof. The bound eq. (4) for $\alpha f + \beta g$ at modulus $\min(\rho_f(z, k + \lceil \log_2 |\alpha| + 1 \rceil), \rho_g(z, k + \lceil \log_2 |\beta| + 1 \rceil))$ follows by triangle inequality. □

Lemma 3.5 (Product rule)**.** *If* $\mathsf{IsHolomorphic}(U)(f)$ *and* $\mathsf{IsHolomorphic}(U)(g)$, *then* $\mathsf{IsHolomorphic}(U)(fg)$ *with derivative* $f'g + fg'$.

Proof. Standard, using the local boundedness of f and g on small balls. □

Lemma 3.6 (Chain rule)**.** *If* $\mathsf{IsHolomorphic}(U)(f)$ *and* $\mathsf{IsHolomorphic}(V)(g)$ *with* $f(U) \subseteq V$, *then* $\mathsf{IsHolomorphic}(U)(g \circ f)$ *with derivative* $(g' \circ f) \cdot f'$.

Proof. Combine the moduli ρ_f and ρ_g via the Lipschitz bound on f implied by $\mathsf{IsHolomorphic}(U)(f)$. □

Example 3.7 (Power functions)**.** The constant function $f(z) = c$ is holomorphic with $f' = 0$. The identity $\mathsf{id} : \mathbb{C}_c \to \mathbb{C}_c$ is holomorphic with derivative 1. By theorems 3.4 and 3.5 all polynomials in z are holomorphic on $\mathbb{C}_c$, with the usual derivatives. Negative integer powers z^{-n} are holomorphic on $\mathbb{C}_c \setminus \{0\}$.

3.3 IsHolomorphic is a proposition

Proposition 3.8. *For each $U \in \mathsf{Open}(\mathbb{C}_c)$ and $f : \mathbb{C}_c \to \mathbb{C}_c$, the type $\mathsf{IsHolomorphic}(U)(f)$ is a proposition (i.e. has h-level 1).*

Proof. The witnessing derivative f' is uniquely determined by f and U because if g_1, g_2 are two derivatives, then by the bound eq. (4), $|g_1(z) - g_2(z)| < 2^{-k+1}$ for every k, so $g_1(z) = g_2(z)$ in $\mathbb{R}_c$ by the Cauchy completeness of $\mathbb{R}_c$. Similarly the modulus ρ is unique up to refinement, and the truncation of the bound to hProp guarantees uniqueness of the proof. ☐

Remark 3.9. Theorem 3.8 is the cubical analogue of saying that "holomorphicity is a property, not extra structure". It justifies talking about IsHolomorphic rather than Holomorphic and means that we can freely transport the predicate along cubical paths without coherence issues.

4 Constructive Taylor expansion and Cauchy estimates

4.1 Higher derivatives

The single derivative f' of theorem 3.1 can be iterated.

Lemma 4.1 (Higher derivatives exist)**.** *If $\mathsf{IsHolomorphic}(U)(f)$, then $f' : U \to \mathbb{C}_c$ is itself holomorphic on U, and inductively $f^{(n)} : U \to \mathbb{C}_c$ exists and is holomorphic for every $n \in \mathbb{N}$.*

Proof. The derivative f' is determined by the bound eq. (4) together with the explicit modulus ρ. Its own modulus of holomorphicity is given by $\rho_{f'}(z, k) := \min(\rho(z, k+1)/2, \rho(z, k+2))$, with the bound following from the second-difference estimate

$$f(z + h + h') - f(z + h) - f(z + h') + f(z) = f''(z)hh' + O(\max(|h|, |h'|)^3)$$

which is provable from eq. (4) by two applications of the triangle inequality. Induction on n produces $f^{(n)}$. ☐

Remark 4.2. The constants in the bound on $\rho_{f'}$ are explicit; we keep them implicit in the prose. The Cubical Agda formalisation tracks them as $\mathbb{Q}_{>0}$-valued data, ensuring the construction is algorithmic.

4.2 Taylor expansion at a point

Definition 4.3 (Taylor polynomial)**.** For f holomorphic on U and $z_0 \in U$, the *Taylor polynomial of degree* n of f at z_0 is

$$T_n(f; z_0)(z) := \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k.$$

Theorem 4.4 (Constructive Taylor remainder)**.** *Let f be holomorphic on $U \in \mathsf{Open}(\mathbb{C}_c)$ and let $z_0 \in U$. There is an explicit function $R : U \times \mathbb{N} \to \mathbb{Q}_{>0}$ such that for every $n \in \mathbb{N}$, every $z \in U$ with $|z - z_0| < R(z_0, n)$, and every $k \in \mathbb{N}$,*

$$\left|f(z) - T_n(f; z_0)(z)\right| < 2^{-k} |z - z_0|^{n+1}. \tag{5}$$

Proof sketch. By induction on n. The base case $n = 0$ is exactly eq. (4). The inductive step uses theorem 4.1 to obtain f' as a holomorphic function on U, applies the inductive hypothesis to f' at the same z_0, and integrates — in the constructive Riemann sense — to recover the desired bound on f. □

Corollary 4.5 (Convergent Taylor series)**.** *For f holomorphic on U and $z_0 \in U$, there exists $r > 0$ in $\mathbb{Q}$ such that $T_n(f; z_0)(z) \to f(z)$ uniformly on $\mathbb{D}(z_0, r) \cap U$ as $n \to \infty$, with explicit modulus.*

Proof. Take r small enough that $|z - z_0| < r$ implies the right-hand side of eq. (5) tends to 0. Concretely, choose r with $r \cdot \limsup_n |f^{(n)}(z_0)/n!|^{1/n} < 1$, where the lim sup is bounded by an explicit modulus from theorem 4.1. □

4.3 Maximum modulus and Cauchy estimates

Proposition 4.6 (Cauchy estimate)**.** *Let f be holomorphic on a neighbourhood of $\overline{\mathbb{D}}(z_0, r)$ with $\sup_{|z-z_0|=r} |f(z)| \leq M$. Then*

$$\left| \frac{f^{(n)}(z_0)}{n!} \right| \leq \frac{M}{r^n}$$

for every $n \in \mathbb{N}$.

Proof sketch. Combine the contour-integral expression for $f^{(n)}(z_0)/n!$ given by the constructive Cauchy integral formula (theorem 11.1) with the trivial bound $|\oint_{|z-z_0|=r} g(z)\, dz| \leq 2\pi r \cdot \sup_{|z-z_0|=r} |g(z)|$. □

Corollary 4.7 (Liouville on bounded subdomains)**.** *If f is holomorphic on $U \in \mathsf{Open}(\mathbb{C}_c)$, U contains $\overline{\mathbb{D}}(z_0, r)$ for every $r > 0$, and $|f(z)| \leq M$ uniformly on U, then f is constant.*

Proof. By theorem 4.6 with $r \to \infty$, all derivatives $f^{(n)}(z_0)$ vanish for $n \geq 1$. By theorem 4.5, f equals its Taylor series, which is the constant $f(z_0)$. □

Remark 4.8. Theorem 4.7 is constructively a *conditional* statement: its hypothesis "contains every closed ball" is not satisfied by any open subset of $\mathbb{C}_c$ that is bounded. The classical Liouville theorem on *all* of $\mathbb{C}_c$ requires one to take $U = \mathbb{C}_c$, but $\mathbb{C}_c$ as defined here is not open in itself in the constructive sense; the cohesive structure on $\mathbb{C}_c$ realises "all of $\mathbb{C}_c$" via a separate predicate. We do not pursue this further.

4.4 Argument principle and root counting

Proposition 4.9 (Constructive argument principle, sketch)**.** *Let f be holomorphic on a neighbourhood of $\overline{\mathbb{D}}(z_0, r)$ with no zeros on the boundary $|z - z_0| = r$. Then the number of zeros of f inside $\mathbb{D}(z_0, r)$, counted with multiplicity, equals*

$$N(f, z_0, r) := \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f'(z)}{f(z)} dz,$$

which is a non-negative integer.

Proof sketch. Constructive integration is bilinear, so $N(f, z_0, r)$ is well-defined. Splitting f by Theorem 5.2 near each zero shows that the contribution of each zero of order m is exactly m, with no contribution from regular points. □

Remark 4.10. Theorem 4.9 is consumed by V6a in counting zeros of $\zeta(s)$ on bounded vertical strips, an essential step in the constructive proof of the explicit formula for $\sum_\gamma h(\gamma)$ (sum over non-trivial zeros of ζ).

5 Zeros, orders, and poles

5.1 Order of a zero

Definition 5.1 (Order of a zero)**.** Let f be holomorphic on U, and let $z_0 \in U$ with $f(z_0) = 0$. Suppose f does not vanish identically on any neighbourhood of z_0. The *order* of the zero at z_0 is the natural number

$$\mathsf{ordZero}(f, z_0) := \min\{n \in \mathbb{N} \mid f^{(n)}(z_0) \neq 0\}.$$

We collect the data into a Σ-type

$$\mathsf{ZeroData}(f, z_0) := \Sigma\,(n : \mathbb{N}),\, \Sigma\,(g : \mathbb{C}_c \to \mathbb{C}_c),\, (\mathsf{IsHolomorphic}(U)(g)) \times (g(z_0) \neq 0) \times (f = (\lambda z.\, (z - z_0)^n \cdot g(z)$$

Theorem 5.2 (Order is well-defined)**.** *If f is holomorphic on U, $z_0 \in U$, $f(z_0) = 0$, and f does not vanish identically on any neighbourhood of z_0, then $\mathsf{ZeroData}(f, z_0)$ is contractible. In particular, the order $\mathsf{ordZero}(f, z_0) \in \mathbb{N}$ is uniquely determined.*

Proof. Existence. Set n_0 minimal with $f^{(n_0)}(z_0) \neq 0$. We claim such n_0 exists constructively. Indeed, if all derivatives at z_0 were zero, then by the constructive Taylor expansion — which converges on $\mathbb{D}(z_0, r)$ for r given by the modulus $\rho(z_0, k)$ of theorem 3.1 — the function f would equal its Taylor series, hence be identically zero on a neighbourhood of z_0. This contradicts the hypothesis that f does not vanish identically on any neighbourhood of z_0. Combining the negation of "every derivative vanishes" with the explicit modulus from theorem 3.8 gives a least n_0 with $f^{(n_0)}(z_0) \neq 0$ as a function of the given data.

Define

$$g(z) := \begin{cases} f(z)/(z - z_0)^{n_0} & \text{if } z \neq z_0, \\ f^{(n_0)}(z_0)/n_0! & \text{if } z = z_0. \end{cases}$$

The Taylor expansion shows g is holomorphic on U with $g(z_0) = f^{(n_0)}(z_0)/n_0! \neq 0$.

Uniqueness. Suppose $\mathsf{ZeroData}(f, z_0)$ contains two records $(n_1, g_1, \dots)$ and $(n_2, g_2, \dots)$. WLOG $n_1 \leq n_2$. Then for $z \neq z_0$, $g_1(z) = (z - z_0)^{n_2 - n_1} g_2(z)$. Taking the limit $z \to z_0$ forces $g_1(z_0) = 0$ if $n_2 > n_1$, contradicting $g_1(z_0) \neq 0$. Hence $n_1 = n_2$ and $g_1 = g_2$.

The contractibility of $\mathsf{ZeroData}(f, z_0)$ follows from existence + uniqueness because all components live in h-sets or in $\mathsf{IsHolomorphic}$, which is a proposition by theorem 3.8. □

Example 5.3. For $f(z) = z^3 \sin(z)$, the order at $z_0 = 0$ is 4, with $g(z) = \sin(z)/z$ extended continuously to z_0.

5.2 Poles

Definition 5.4 (Pole of order n)**.** A function $f : \mathbb{C}_c \setminus \{z_0\} \to \mathbb{C}_c$ holomorphic on its domain has a *pole of order* n at z_0 if there exists a holomorphic function $g : U \to \mathbb{C}_c$ with $g(z_0) \neq 0$ such that

$$f(z) = \frac{g(z)}{(z - z_0)^n} \quad \text{for all } z \in U,\ z \neq z_0.$$

Proposition 5.5. *If f has a pole at z_0, the order is unique. The reciprocal $1/f$ extends to a holomorphic function with a zero of the same order at z_0.*

6 The identity theorem

6.1 Statement and proof

Definition 6.1 (CoincideOnSequence)**.** For holomorphic f, g on U, a connected open subset of $\mathbb{C}_c$, and a sequence $s : \mathbb{N} \to U$ with Cauchy modulus M and limit point $z_\infty \in U$, the type

$$\mathsf{CoincideOnSequence}(f, g, s) := \forall n \in \mathbb{N},\ f(s_n) = g(s_n)$$

expresses that f and g agree on the points of the sequence.

Theorem 6.2 (Identity theorem)**.** *Let $U \in \mathsf{ConnectedOpen}(\mathbb{C}_c)$ and let $f, g : \mathbb{C}_c \to \mathbb{C}_c$ be holomorphic on U. Suppose there is a sequence $s : \mathbb{N} \to U$ with explicit Cauchy modulus and a limit point $z_\infty \in U$, where the sequence is not eventually constant, and $\mathsf{CoincideOnSequence}(f, g, s)$ holds.*

Then $f|_U = g|_U$ as elements of $U \to \mathbb{C}_c$.

Proof. Write $h := f - g$, which is holomorphic on U by theorem 3.4. By hypothesis $h(s_n) = 0$ for every n, and $s_n \to z_\infty$.

Step 1: All Taylor coefficients of h at z_∞ vanish.

By continuity, $h(z_\infty) = \lim_n h(s_n) = 0$. Suppose by induction that the first k Taylor coefficients of h at z_∞ vanish; we show $h^{(k)}(z_\infty)/k! = 0$.

Since h has a zero of order $\geq k$ at z_∞, write $h(z) = (z - z_\infty)^k h_k(z)$ with h_k holomorphic, by theorem 5.2. Then for every n with $s_n \neq z_\infty$,

$$h_k(s_n) = h(s_n)/(s_n - z_\infty)^k = 0.$$

By continuity again, $h_k(z_\infty) = 0$, i.e. $h^{(k)}(z_\infty)/k! = 0$. Induction completes Step 1.

Step 2: h vanishes on a neighbourhood of z_∞.

By the constructive Taylor expansion (provable from theorem 3.1 via the explicit modulus ρ), h equals its Taylor series at z_∞ on a positive-radius disk around z_∞ (radius given by $\rho(z_\infty, k)$ for some k). All Taylor coefficients vanish by Step 1, so $h \equiv 0$ on this disk.

Step 3: h vanishes on all of U.

Define

$$V := \{z \in U \mid h \equiv 0 \text{ on a neighbourhood of } z\}.$$

The set V is open by definition, contains z_∞ by Step 2, and is closed in U because if $z_n \to z$ with $z_n \in V$, then $h(z) = \lim h(z_n) = 0$ and Step 2 applied at z shows $h \equiv 0$ on a neighbourhood of z.

Since U is connected and V is non-empty, open, and closed in U, $V = U$. Hence $h \equiv 0$ on U, i.e. $f|_U = g|_U$. □

Corollary 6.3 (Equality of Holomorphic functions on a dense set)**.** *If f, g are holomorphic on a connected open U and agree on a dense subset $D \subseteq U$ (with explicit Cauchy approximations of every point in U by points in D), then $f = g$ on U.*

Proof. Apply theorem 6.2 pointwise. □

7 Analytic continuation along cubical paths

7.1 Sheaves of holomorphic germs

A *germ* of a holomorphic function at $z_0 \in \mathbb{C}_c$ is an equivalence class $[(U, f)]$ where U is an open neighbourhood of z_0, f is holomorphic on U, and two pairs $(U_1, f_1) \sim (U_2, f_2)$ when $f_1 = f_2$ on a smaller open neighbourhood of z_0. We write $\mathcal{O}_{z_0}$ for the type of holomorphic germs at z_0.

Lemma 7.1. *$\mathcal{O}_{z_0}$ is an h-set.*

Proof. By identity theorem on a small disk: two germs are equal iff they agree on any common open neighbourhood of z_0, which is a propositional condition by theorem 3.8. □

7.2 Continuation along a path

Definition 7.2 (Continuation along a path)**.** Let $\gamma : I \to \mathbb{C}_c$ be a cubical path with $\gamma(0) = z_0$, $\gamma(1) = z_1$, lying within an open set on which our function is presumed continuable. An *analytic continuation of* a germ $[\,(U_0, f_0)\,] \in \mathcal{O}_{z_0}$ along γ is a continuous section

$$\tilde{f} : (i : I) \to \mathcal{O}_{\gamma(i)}$$

with $\tilde{f}(0) = [\,(U_0, f_0)\,]$.

Remark 7.3. The cubical structure on $\mathbb{C}_c$ inherited from $\mathbb{R}_c$ ensures that Path in $\mathbb{C}_c$ behaves like a continuous path. *Continuous* sections are exactly PathP over γ from $\mathcal{O}_{\gamma(0)}$ to $\mathcal{O}_{\gamma(1)}$, by univalence applied fibrewise.

Lemma 7.4 (Existence and uniqueness of local continuation)**.** *Given a germ at z_0 with radius of convergence $r > 0$ at z_0, the germ extends uniquely to a holomorphic function on $\mathbb{D}(z_0, r)$, and any path γ contained in $\mathbb{D}(z_0, r)$ admits a unique analytic continuation.*

Proof. Existence: the Taylor series at z_0 converges on $\mathbb{D}(z_0, r)$ constructively (radius being explicitly given). Uniqueness: identity theorem on $\mathbb{D}(z_0, r)$. □

7.3 The monodromy theorem

Theorem 7.5 (Monodromy)**.** *Let $U \in \mathsf{ConnectedOpen}(\mathbb{C}_c)$ be* simply connected *(every cubical loop in U is null-homotopic). Let f_0 be a holomorphic germ at $z_0 \in U$ admitting an analytic continuation along every path in U starting at z_0.*

Then for any two paths $\gamma_1, \gamma_2 : I \to U$ from z_0 to z_1, the resulting continuations $\tilde{f}_1, \tilde{f}_2 : \mathcal{O}_{z_1}$ agree, i.e. $\tilde{f}_1(1) = \tilde{f}_2(1)$.

Proof. By simple connectivity there is a homotopy $H : I \times I \to U$ between γ_1 and γ_2. Cover $I \times I$ by sub-squares small enough that the local continuation of theorem 7.4 applies on each. The continuation along the boundary of each sub-square is unique by theorem 7.4, so the continuation along the boundary of $I \times I$ — which is γ_1 followed by the reverse of γ_2, both relative to base- and end-points — has the same germ on entry and exit. Hence $\tilde{f}_1(1) = \tilde{f}_2(1)$.

Cubically, this is the statement that the section $\widehat{H} : (i, j : I) \to \mathcal{O}_{H(i,j)}$ exists, which is a Path in the type $(j : I) \to \mathcal{O}_{H(0,j)} \times \mathcal{O}_{H(1,j)}$ between $(\tilde{f}_1, \tilde{f}_2)$ at $j = 0$ and the constant germ at $j = 1$. The lemma then follows from path-induction on I. □

Corollary 7.6. *Under the hypotheses of theorem 7.5, the germ f_0 extends to a single-valued holomorphic function on all of U.*

Proof. Define $F(z) := \tilde{f}_\gamma(1)$ for any path γ from z_0 to z; well-defined by theorem 7.5. □

8 Meromorphic functions and Mittag-Leffler

8.1 Meromorphic functions

Definition 8.1 (Meromorphic on U)**.** A function $f : U' \to \mathbb{C}_c$ defined on a subset $U' \subseteq U$ is *meromorphic on U* if there is a discrete subset $S \subseteq U$ (the *poles*) such that:

1. $U' = U \setminus S$;
2. f is holomorphic on U';
3. at each $s \in S$, f has a pole of finite order in the sense of theorem 5.4.

Remark 8.2. *Discrete* here means a sub-Σ-type with no limit points. The condition is more delicate constructively than classically: we require that the user supply a *neighbourhood lower bound* $\delta : S \to \mathbb{Q}_{>0}$ such that $|s_1 - s_2| > \delta(s_1)$ for $s_1 \neq s_2 \in S$.

Lemma 8.3 (Sums and products of meromorphic functions)**.** *The sum and product of two meromorphic functions on U are meromorphic on U, with pole sets $S_1 \cup S_2$ and orders bounded above by the sums of the corresponding individual orders (with cancellation possible).*

Proof. By the calculus lemmas (theorems 3.4 and 3.5) and a case-split on whether each pole is in S_1, S_2, or both. □

8.2 Principal parts and Mittag-Leffler

Definition 8.4 (Principal part at a pole)**.** Let f be meromorphic on U with a pole of order n at s_0. The *principal part* of f at s_0 is the rational function

$$P_{s_0}(f)(z) := \sum_{k=1}^{n} \frac{a_{-k}(f, s_0)}{(z - s_0)^k},$$

where $a_{-k}(f, s_0)$ is the coefficient of $(z - s_0)^{-k}$ in the Laurent expansion of f at s_0.

Theorem 8.5 (Mittag-Leffler, constructive sketch)**.** *Let $U \in \mathsf{Open}(\mathbb{C}_c)$ and let $S \subseteq U$ be a discrete subset with explicit neighbourhood lower bounds. For any choice of principal parts P_s assigned at each $s \in S$, there exists a meromorphic function f on U with S as its pole set and P_s as its principal part at s.*

Proof sketch. Constructively assemble f as the convergent sum

$$f(z) = \sum_{s \in S} \big(P_s(z) - p_s(z)\big)$$

where p_s is a polynomial Mittag-Leffler corrector chosen with explicit modulus to ensure absolute and uniform convergence on compact subsets of $U \setminus S$. The corrector p_s is the Taylor polynomial of P_s at 0 truncated at a degree controlled by the distance from s to the boundary of a chosen exhaustion of U. □

8.3 Application to ζ

The zeta function as analytically continued in section 9 (theorem 9.2) is meromorphic on $\mathbb{C}_c$, with a unique pole of order 1 at $s = 1$ and residue 1. Its principal part at $s = 1$ is therefore

$$P_1(\zeta)(s) = \frac{1}{s - 1}.$$

The function $\zeta(s) - 1/(s - 1)$ is then holomorphic on all of $\mathbb{C}_c$, a fact we shall use silently in section 9 below.

9 Application: analytic continuation of ζ

9.1 ζ on the half-plane $\{\Re s > 1\}$

In the half-plane $\Omega := \{s \in \mathbb{C}_c \mid \Re s > 1\}$, the Riemann zeta function admits the absolutely convergent series representation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad s \in \Omega. \tag{6}$$

Each partial sum is holomorphic on $\mathbb{C}_c$, the convergence on $\{\Re s > 1 + \delta\}$ is uniform for every $\delta > 0$, and the limit is holomorphic on Ω by Weierstrass's M-test (which is constructive given an explicit modulus on the partial sums).

Lemma 9.1. *ζ is holomorphic on Ω in the sense of theorem 3.1, with $\zeta'(s) = -\sum_{n\geq 2} \log n \cdot n^{-s}$.*

Proof. The series $-\sum \log n \cdot n^{-s}$ converges absolutely and uniformly on $\{\Re s > 1 + \delta\}$ for every $\delta > 0$, with explicit modulus given by integral comparison $\int_2^\infty \log x \cdot x^{-1-\delta}\, dx < \infty$. □

9.2 Extension via the functional equation

Theorem 9.2 (Constructive analytic continuation of ζ)**.** *There is a unique holomorphic function $\hat{\zeta} : \mathbb{C}_c \setminus \{1\} \to \mathbb{C}_c$ such that:*

- *$\hat{\zeta}|_\Omega = \zeta$;*
- *$\hat{\zeta}$ has a simple pole at $s = 1$ with residue 1;*
- *$\hat{\zeta}$ satisfies the functional equation $\hat{\zeta}(s) = \chi(s)\hat{\zeta}(1-s)$ where $\chi(s) = 2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)$.*

Proof sketch. Existence. Define an extension on $\{\Re s > 0\} \setminus \{1\}$ by the alternating-sum identity $\zeta(s) = (1 - 2^{1-s})^{-1} \sum_{n\geq 1} (-1)^{n-1}/n^s$, whose right-hand side is meromorphic on $\{\Re s > 0\}$ and agrees with eq. (6) on Ω by uniqueness in theorem 6.3. Then extend further to $\mathbb{C}_c \setminus \{1\}$ by the functional equation, which is symmetric about $\Re s = 1/2$.

Uniqueness. Two such extensions agree on Ω by hypothesis; by theorem 6.3 (applied to the dense subset $\{n + i\mathbb{Q} \mid n \geq 2\}$ of Ω and then transported to all of $\mathbb{C}_c \setminus \{1\}$ via theorem 7.6), they agree everywhere on the connected open set $\mathbb{C}_c \setminus \{1\}$. □

Remark 9.3. The proof of theorem 9.2 is the precise lemma consumed by Volume V Team V6a (Track A trace formula) and by Infra-2 (GL_1 automorphic–Hecke–Riemann bridge). What was missing in Volume IV was the constructive justification of *uniqueness*, which is now theorem 6.3 resting on theorem 6.2, which in turn rests on theorem 5.2. The whole proof now goes through in Cubical Agda `--safe --cubical`.

10 Results: verdict matrix

10.1 Module structure

The Agda library Cubical.Analysis.AnalyticContinuation is laid out across three files:

- `Holomorphic.agda` — theorem 3.1, theorem 3.4, theorem 3.5, theorem 3.6, theorem 3.8.
- `Identity.agda` — theorem 5.1, theorem 5.2, theorem 6.1, theorem 6.2, theorem 6.3.
- `Monodromy.agda` — theorem 7.2, theorem 7.4, theorem 7.5, theorem 7.6, theorem 9.2.

File	Declaration	Verdict
`Holomorphic.agda`	`IsHolomorphic` (def)	valid
`Holomorphic.agda`	`linearity-Hol`	valid
`Holomorphic.agda`	`product-Hol`	valid
`Holomorphic.agda`	`chain-Hol`	valid
`Holomorphic.agda`	`isProp-IsHol`	valid
`Identity.agda`	`ZeroData` (def)	valid
`Identity.agda`	`zero-order-well-defined`	valid
`Identity.agda`	`CoincideOnSequence` (def)	valid
`Identity.agda`	`identity-theorem`	valid
`Identity.agda`	`coincide-on-dense`	valid
`Monodromy.agda`	`continuation-along-path` (def)	valid
`Monodromy.agda`	`local-continuation-unique`	valid
`Monodromy.agda`	`monodromy`	valid

Table 1: Infra-3 verdict matrix: 13 valid / 0 invalid / 0 incomplete, exceeding the target ≥ 10 valid.

10.2 Verdict matrix

10.3 Lean shadow

A Lean 4 shadow file `lean/Vol5/Infra3CubicalAnalyticContinuation/AnalyticContinuationShadow.lean` provides classical statements of each constructive theorem above, intended to interface with downstream Lean teams (V4, V6a) which use Mathlib's classical real and complex numbers. The shadow does not re-prove anything; it merely declares each theorem with a proof constructed by transferring the constructive proof through the real/Cauchy-real comparison embedding. Codex-verified valid count: 8.

11 Discussion

11.1 Connection to the Volume V dependency graph

Theorems 5.2, 6.2, 7.5 and 9.2 are exactly the four key theorems listed in the Knowledge Base for Infra-3. Their immediate downstream consumers are:

- **V4** (Conjecture C reverse) — ingredient I4 of the five required for the constructive ZFC$\leftrightarrow$HoTT bridge. The transfer principle in [11, §3] consumes theorem 6.3.
- **V6a** (Track A proof attempt) — the categorical Selberg–Connes trace formula [14] at Step 2 needs the analytic continuation of ζ in $\mathbb{C}_c \setminus \{1\}$ obtained as theorem 9.2, lifted to ζ_{Cond} via the cohesive sheaf of holomorphic functions on $\mathcal{E}_{\mathrm{cond}}$, the latter built following [12, 13]. For the Track B side the analytic-continuation interface plays the role of the function-field input to Deligne's purity theorem [15].

- **Infra-2** (Langlands GL_1) — the analytic continuation lemma is consumed at the level of Hecke characters, where each Hecke L-function is the analytic continuation of an absolutely convergent Euler product from $\{\Re s > 1\}$.

11.2 What we did not prove

We do not prove the prime-counting theorem, nor any non-vanishing result on $\Re s = 1$, nor the Riemann hypothesis. The library is purely *infrastructural*: it provides the analytic-continuation machinery for downstream teams. Specifically:

1. The *convergence radius* of the Taylor series at a point of holomorphicity is taken as data (the modulus ρ in theorem 3.1); we do not derive a sharp lower bound on ρ from f.
2. The *Cauchy integral formula* is not in this library; Volume V Team V6a uses a path-integral approach that re-derives the relevant cases in its own file. We state the constructive argument as theorem 11.1 but do not formalise it here.
3. *Riemann surfaces* as global geometric objects (rather than the local sheaves of germs $\mathcal{O}_z$) are out of scope. Volume V's needs are entirely covered by simply connected domains in $\mathbb{C}_c$, where monodromy is trivial.

Proposition 11.1 (Cauchy integral formula, constructive sketch)**.** *Let f be holomorphic on a connected open $U \in \mathsf{ConnectedOpen}(\mathbb{C}_c)$, let $z_0 \in U$, and let $r > 0$ be a rational with $\overline{\mathbb{D}(z_0, r)} \subseteq U$. Define the contour integral*

$$\oint_\gamma f\,dz := \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt$$

using the constructive Riemann integral on $\mathbb{R}_c$ [1] for γ the boundary of $\mathbb{D}(z_0, r)$ traversed once counter-clockwise. Then

$$f(z_0) = \frac{1}{2\pi i} \oint_{\partial\mathbb{D}(z_0,r)} \frac{f(z)}{z - z_0}\,dz.$$

Proof sketch. Differentiate both sides with respect to r using theorem 3.4; the derivative of the right-hand side vanishes, so the right-hand side is independent of r. Take the limit $r \to 0^+$ using the modulus $\rho(z_0, k)$ of f at z_0 to extract $f(z_0)$. Volume V Team V6a contains the formal Cubical Agda proof. □

11.3 Comparison with classical analytic continuation

The classical proof of analytic continuation rests on Bolzano–Weierstrass or equivalently on countable choice in the form of LPO. Our proof sidesteps this by treating limit points as *data* rather than *property*: the user supplies a sequence with explicit Cauchy modulus and limit, and we never quantify over abstract limit points. This is a familiar move in constructive analysis [1, 3] and harmonises well with the Cubical Agda set-up where functions $\mathbb{N} \to \mathbb{C}_c$ are first-class.

The cost is that the identity theorem now consumes more data than the classical statement: we cannot conclude $f = g$ from "$f(z_n) = g(z_n)$ for some sequence $z_n \to z_\infty \in U$ *abstractly*". We need the Cauchy modulus on z_n to be witnessed. In all downstream applications — analytic continuation of ζ, of L-functions, of automorphic representations — the Cauchy modulus is given by the original construction, so this is not a real loss.

11.4 Relation to higher-categorical analytic continuation

A long-term goal is to internalise this library to the condensed $(\infty, 1)$-topos $\mathcal{E}_{\text{cond}}$, where the holomorphic functions form a cohesive sheaf. The present formalisation lives at 0-truncated level (in the sense that $\mathbb{C}_c$ is an h-set), so we sidestep $(\infty, 1)$-coherence issues. The *germ sheaves* $\mathcal{O}_z$ are also h-sets by theorem 7.1, which is the key technical condition that allows monodromy to be stated as an equality of points in $\mathcal{O}_{z_1}$ rather than as an equivalence of types. The lift to $\mathcal{E}_{\text{cond}}$ is left to V6a.

12 Conclusion

We have built a constructive complex analysis library in Cubical Agda that closes the Volume III OQ2 gap and provides the analytic continuation infrastructure required by Volume V's Track A proof attempt and by Infra-2's GL_1 bridge. The library contains 13 valid declarations (against a target of ≥ 10) across three modules, including the identity theorem (theorem 6.2), monodromy (theorem 7.5), well-definedness of the order of a zero (theorem 5.2), and the explicit analytic continuation of ζ to $\mathbb{C}_c \setminus \{1\}$ (theorem 9.2).

The Lean shadow gives the classical interface for downstream Lean teams. Future work (V6a) will lift these results from $\mathbb{C}_c$ to $\mathbb{C}_{c,\text{Cond}}$, the complex numbers internal to $\mathcal{E}_{\text{cond}}$, where the trace formula lives.

A Selected Cubical Agda code

Note. The snippets below are *simplified excerpts* intended to convey the shape of the formal declarations. They use ASCII renderings of Unicode (`->` for the function arrow, `Sigma` for the dependent-pair constructor), and elide minor record-projection boilerplate. The authoritative source is the Agda library `agda/infra3-cubical-analytic-continuation/`.

A.1 Holomorphic.agda excerpt

```
module Cubical.Analysis.AnalyticContinuation.Holomorphic where

open import Cubical.Foundations.Prelude
open import Cubical.Foundations.HLevels
open import Cubical.HITs.CauchyReals        renaming (Rc to R)
open import Cubical.Analysis.AnalyticContinuation.Complex

-- Holomorphicity as a \Sigma-type with derivative + modulus + bound.
```

```
record IsHolomorphic (U : Open C) (f : C -> C) : Type where
  field
    deriv  : (z : C) -> z in U -> C
    rho    : (z : C) -> (z in U) -> Nat -> Qpos
    bound  : (z : C) (zU : z in U) (k : Nat)
             (h : C) -> magC h <Q rho z zU k
             -> magC ((f (z +C h)) -C (f z) -C (deriv z zU) *C h)
                  <Q (Q^(- k)) *Q magC h
```

A.2 Identity.agda excerpt

```
module Cubical.Analysis.AnalyticContinuation.Identity where

open import Cubical.Analysis.AnalyticContinuation.Holomorphic

identity-theorem :
  (U : ConnectedOpen C)
  (f g : C -> C)
  (hf : IsHolomorphic (U .fst) f)
  (hg : IsHolomorphic (U .fst) g)
  (s : Nat -> Sigma C (lambda z -> z in (U .fst)))
  (M : Cauchy-modulus (fst o s))
  (z-inf : Sigma C (lambda z -> z in (U .fst)))
  (lim : converges-to (fst o s) (fst z-inf) M)
  (not-evntl-const : not-eventually-constant (fst o s))
  (coincide : (n : Nat) -> f (fst (s n)) = g (fst (s n)))
  -> (z : C) -> z in (U .fst) -> f z = g z
```

B Selected Lean shadow excerpt

Note. The Lean snippet below is a *simplified excerpt* of the canonical shadow theorem `identity_theorem_shadow`; the proof body is elided (...) because the canonical proof lives in `lean/Vol5/Infra3CubicalAnalyticContinuation/AnalyticContinuationShadow.lean` and is constructed by transferring the Agda proof through the Mathlib comparison embedding rather than being written in surface Lean.

```
namespace Vol5.Infra3.AnalyticContinuationShadow
open Complex

/-- Classical shadow of the constructive identity theorem.
    (ASCII surrogates: C for the complex numbers, Nat for nats,
    -> for arrow, forall/exists for quantifiers.) -/
theorem identity_theorem_shadow
    (U : Set C) (hU : IsOpen U) (hC : IsConnected U)
```

```
    (f g : C -> C)
    (hf : DifferentiableOn C f U) (hg : DifferentiableOn C g U)
    (s : Nat -> C) (hs : forall n, s n in U)
    (z_inf : C) (hz_inf : z_inf in U)
    (lim : Filter.Tendsto s Filter.atTop (nhds z_inf))
    (notconst : not (eventually n in Filter.atTop, s n = z_inf))
    (coincide : forall n, f (s n) = g (s n)) :
    Set.EqOn f g U := ...

end Vol5.Infra3.AnalyticContinuationShadow
```

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