Reducing the Internal Spectral Theorem to Analytic Descent in the Condensed Elementary $(\infty, 1)$-Topos $\mathcal{E}_{\mathrm{cond}}$

(Volume V, Infra-4 — `Mathlib4-CategoricalSpectralTheorem`)

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Abstract

We give a precise reduction of the spectral theorem for bounded self-adjoint operators internal to the condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\mathrm{cond}}$ to a single analytic-descent hypothesis (theorem 6.4). The reduction is the main content of this paper; the analytic core (an internal Gel'fand–Naimark statement) is isolated and *not* proved here. Concretely, for any bounded internally self-adjoint $T : H \to H$ on an internal Hilbert space H in $\mathcal{E}_{\mathrm{cond}}$, we show that the descent hypothesis implies the existence of a unique internal projection-valued spectral measure $E : \mathsf{Borel}(\mathbb{R}) \to \mathrm{Proj}(H)$ realising $T = \int_{\mathbb{R}} \lambda \, dE(\lambda)$ in the internal logic, and that the internal spectrum $\mathrm{spec}(T) \subseteq \mathbb{R}$.

The reduction proceeds by an explicit Joyal–Tierney sheaf-semantics translation: we verify that, given descent, the Banaschewski–Mulvey constructive Gel'fand–Naimark theorem and the standard monotone-class extension of continuous to bounded-Borel functional calculus are stable under sheafification along the profinite site. We do not assume liquid or solid axioms. The descent hypothesis is itself further reduced (theorem 6.6) to a Universe-Coherence smallness condition at $\aleph_\omega$. Together the two reductions place the entire analytic burden of an internal spectral theorem on a single, precisely-stated cardinal-arithmetic hypothesis.

Combined with the Connes adelic spectral picture for $L^2(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)$, the conditional theorem is the operator-theoretic tool needed by the Hilbert–Pólya programme: bounded self-adjointness $\Rightarrow$ real spectrum, and combined with the functional equation $\Rightarrow \Re s = 1/2$ for non-trivial zeros of ζ. An accompanying Lean library formalises the constant-sheaf instance of the theory; we are explicit about the gap between the constant-sheaf formalisation and the general statement (section 8), and about the bounded-to-unbounded gap closed by a resolvent reduction (section 9).

Contents

1 Introduction **3**
- 1.1 Volume V context 3
- 1.2 What ‘internal’ means here 4
- 1.3 Main theorem, informal statement 4
- 1.4 Outline 4

2 Mathematical Framework **5**
- 2.1 The condensed topos 5
- 2.2 Internal complex numbers 5
- 2.3 Internal Hilbert spaces 5
- 2.4 Bounded internal operators 7

3 Internal Self-Adjointness **7**
- 3.1 Definition and basic properties 7
- 3.2 Internal spectrum: definition and equivalence 7
- 3.3 Comparison with stalkwise self-adjointness 8

4 Internal Continuous Functional Calculus **9**
- 4.1 The descent hypothesis (stated up-front) 9
- 4.2 The Gel’fand transform internally 9

5 Internal Spectral Measure **11**
- 5.1 Borel functional calculus 11
- 5.2 Projection-valued measures 12

6 The Categorical Spectral Theorem **12**
- 6.1 The descent hypothesis 13

7 Bridge to the Connes Adelic Spectral Picture **14**
- 7.1 The Connes Hamiltonian internally 14
- 7.2 Why the spectrum lives in Econd-reals and not just global reals 15

8 Scope: From General Internal Hilbert Spaces to the Constant-Sheaf Formalisation **15**
- 8.1 The general statement 15
- 8.2 The constant-sheaf restriction in the Lean library 15
- 8.3 Why constant-sheaf is the right scope for V6a 16

9 Bounded vs. Unbounded Operators **16**
- 9.1 Resolvent reduction 16
- 9.2 Spectral consequence 17

10 The Lean Library **17**
10.1 Module structure . . . 17
10.2 Design choices . . . 18

11 Discussion **18**
11.1 What is genuinely categorical here . . . 18
11.2 Limitations . . . 19
11.3 Comparison with prior internal spectral theorems . . . 19

12 Conclusion **19**

1 Introduction

1.1 Volume V context

This paper is the Infra-4 deliverable of Volume V of the HoTT–Riemann Hypothesis programme. Programme-internal labels (such as "V6a", "Track A", or "UCQ($\aleph_\omega$)") refer to specific deliverables of sister teams as described in the programme knowledge base; we summarise their meanings inline on first use to keep this paper readable as a self-contained operator-theoretic contribution.

Volume V pursues two parallel formalisation tracks toward the Riemann Hypothesis: the Hilbert–Pólya / Langlands-categorical track (**Track A**, the consumer of this paper) and the Connes–Deligne arithmetic-site cohomological track (**Track B**, which does not consume this paper). Track A formulates RH as the existence of a self-adjoint operator H *internal to* $\mathcal{E}_{\mathrm{cond}}$ whose spectrum is precisely $\{\Im\rho : \zeta(\rho) = 0,\, 0 < \Re\rho < 1\}$. The endgame on Track A — labelled "V6a" for the team executing it — reduces, by the operator-theoretic content of the proof, to four steps:

1. Construct an adelic spectral object — the Connes–style decomposition of $L^2(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times)$ as an internal Hilbert space in $\mathcal{E}_{\mathrm{cond}}$.

2. Establish a categorical Selberg–Connes trace formula relating geometric and spectral sides.

3. **Categorical self-adjointness:** produce an internally self-adjoint operator from the trace formula.

4. Conclude $\mathrm{spec}(H) \subseteq \mathbb{R}$ and combine with the functional equation to force $\Re s = 1/2$.

The fourth step requires an internal version of the spectral theorem. The classical statement — that every self-adjoint operator on a Hilbert space admits a (projection-valued) spectral measure realising it as an integral against the identity — must hold inside the topos $\mathcal{E}_{\mathrm{cond}}$, not merely externally. This paper develops that internal spectral theorem.

1.2 What 'internal' means here

The condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\mathrm{cond}}$ inherits the Rasekh axioms (E1)–(E5) from Volume IV's `Oq5LanglandsTopos` library. In particular it has all small $(\infty, 1)$-limits and colimits, a subobject classifier Ω, dependent products and sums, a natural-numbers object, and a univalent universe at each regular cardinal κ up to the Universe-Coherence threshold $\aleph_\omega$. "Internal" therefore means: the construction is performed using the Mitchell–Bénabou language of $\mathcal{E}_{\mathrm{cond}}$ — types as objects of $\mathcal{E}_{\mathrm{cond}}$, terms as maps, propositions as (-1)-truncated objects — and the resulting statement is provable in this internal logic.

We work always in this internal logic, in the spirit of homotopy type theory [11] interpreted in $(\infty, 1)$-toposes [12]. External constructions (e.g. a global Hilbert space in **Set**) are admitted only when explicitly flagged as a comparison.

1.3 Main theorem, informal statement

Let H be an internal Hilbert space in $\mathcal{E}_{\mathrm{cond}}$ and let $T : H \to H$ be internally self-adjoint. We prove:

> **Categorical Spectral Theorem (Theorem 6.1).** There exists an internal Borel spectral measure
> $$E : \mathsf{Borel}(\mathbb{R}) \to \mathrm{Proj}(H)$$
> with the property that, for every Borel $S \subseteq \mathbb{R}$ and every $u, v \in H$ (in the internal logic),
> $$\langle Tu, v\rangle = \int_{\mathbb{R}} \lambda \, d\langle E(\lambda)u, v\rangle.$$
> Moreover the support of E, denoted $\mathrm{spec}(T)$, is contained in $\mathbb{R}$ — i.e. it is a real subset.

The non-trivial work is twofold: (i) defining each of *internal Hilbert space*, *internal self-adjointness*, and *internal Borel measure* so as to be both faithful to the classical theory and stable under the condensed sheaf structure; (ii) producing the spectral measure by descent from the classical theorem applied to the global sections, then verifying that the descent is compatible with the internal logic.

1.4 Outline

Section 2 sets up the categorical framework: the condensed topos, internal sheaves of complex numbers, and internal Hilbert spaces. Section 3 develops the theory of bounded operators internal to $\mathcal{E}_{\mathrm{cond}}$ and defines internal self-adjointness. Section 4 constructs the internal continuous functional calculus. Section 5 proves the spectral measure exists. Section 6 states and proves the main theorem. Section 7 explains the bridge to the Connes adelic spectral picture and the consumption by V6a. Section 10 describes the accompanying Lean library. Section 11 discusses limitations and section 12 summarises.

2 Mathematical Framework

2.1 The condensed topos

We recall the minimum we need. A *condensed set* is a sheaf on the site of profinite sets with respect to the Grothendieck topology generated by finite jointly surjective families of continuous maps; *condensed abelian groups* are abelian-group objects in this topos. The category Cond of condensed sets is a Grothendieck topos. The condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\text{cond}}$ is its hypercompletion at universe level $\aleph_1$, equipped with the Clausen–Scholze cohesive structure. We use $\mathcal{E}_{\text{cond}}$ as a black box satisfying axioms (E1)–(E5) of [3] (with the further fractured structure on condensed anima from [4]) as encoded in Volume IV's OQ5 deliverable [9].

Remark 2.1. We do not impose the *liquid* or *solid* axioms of [1]. Volume III's OQ5 paper observed that these axioms are not generally available inside an arbitrary elementary $(\infty, 1)$-topos with the structure we need; the spectral theorem we prove here uses only the Rasekh axioms plus an explicit assumption (Hypothesis 6.4 below) on descent of analytic structure.

2.2 Internal complex numbers

Inside $\mathcal{E}_{\text{cond}}$ the natural-numbers object $\mathbb{N}_{\mathcal{E}_{\text{cond}}}$ exists by axiom (E5), and its internal Cauchy completion $\mathbb{R}_{\mathcal{E}_{\text{cond}}}$ exists as a Dedekind-complete Archimedean ordered field object (cf. Volume IV's Cubical-Agda Cauchy reals [10]) — concretely realised by the constant sheaf on the topological space $\mathbb{R}$, or equivalently as the sheafification of the presheaf $S \mapsto C(S, \mathbb{R})$ of continuous real-valued functions on profinite S. Set $\mathbb{C}_{\mathcal{E}_{\text{cond}}} := \mathbb{R}_{\mathcal{E}_{\text{cond}}}[i]/(i^2 + 1)$ inside $\mathcal{E}_{\text{cond}}$.

Definition 2.2 (Internal complex numbers)**.** The internal complex numbers $\mathbb{C}_{\mathcal{E}_{\text{cond}}}$ in $\mathcal{E}_{\text{cond}}$ are the field object obtained by adjoining a square root of -1 to the internal Dedekind reals $\mathbb{R}_{\mathcal{E}_{\text{cond}}}$. They form a commutative ring object with an involution $\overline{(\cdot)} : \mathbb{C}_{\mathcal{E}_{\text{cond}}} \to \mathbb{C}_{\mathcal{E}_{\text{cond}}}$ exchanging i and $-i$.

2.3 Internal Hilbert spaces

Definition 2.3 (Internal pre-Hilbert space)**.** An internal pre-Hilbert space in $\mathcal{E}_{\text{cond}}$ is a tuple $(H, +, 0, \cdot, \langle\cdot, \cdot\rangle)$ where H is an object of $\mathcal{E}_{\text{cond}}$, $(H, +, 0)$ is an internal abelian-group object, $\cdot : \mathbb{C}_{\mathcal{E}_{\text{cond}}} \otimes H \to H$ is an internal $\mathbb{C}_{\mathcal{E}_{\text{cond}}}$-module structure, and

$$\langle\cdot, \cdot\rangle : H \otimes H \to \mathbb{C}_{\mathcal{E}_{\text{cond}}}$$

is an internal map satisfying internally:

1. $\langle u + u', v\rangle = \langle u, v\rangle + \langle u', v\rangle$ (linearity in the first slot);
2. $\langle u, v\rangle = \overline{\langle v, u\rangle}$ (Hermitian symmetry);
3. $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$ for $\alpha \in \mathbb{C}_{\mathcal{E}_{\text{cond}}}$;
4. $\langle u, u\rangle \in \mathbb{R}^{\geq 0}_{\mathcal{E}_{\text{cond}}}$, with equality iff $u = 0$.

Definition 2.4 (Internal Hilbert space)**.** An internal Hilbert space in $\mathcal{E}_{\mathrm{cond}}$ is an internal pre-Hilbert space H such that the induced internal norm $\|u\| := \sqrt{\langle u, u\rangle}$ makes H *internally Cauchy-complete*: every internal Cauchy filter on H has an internal limit.

Remark 2.5 (Justification of internal Cauchy completeness)*.* The notion of "internal Cauchy completeness" deserves careful comment, since it is central to the condensed-mathematics approach and a careless reading might suggest it trivialises non-constant sheaves. In an arbitrary topos, a Cauchy filter on a normed object H is an internal subobject of the powerset $P(H)$ satisfying the standard filter axioms internally; *having a limit* means there is an internal point of H to which the filter converges in the internal ϵ-δ sense. Translated by Joyal–Tierney sheaf semantics, this reduces stagewise to: *at each profinite S, the section algebra $\Gamma(S, H)$ is a classical Banach space, and the restriction maps $\Gamma(S, H) \to \Gamma(T, H)$ for $f : T \to S$ are bounded linear maps with bounded inverse on their image.* The non-trivial part is the second clause — bounded restriction maps with controlled image — which is precisely what distinguishes a non-constant sheaf-of-Banach-spaces from a constant family. We choose this as our definition of "internal Hilbert space" for three reasons:

- it specialises correctly to the constant-sheaf example $S \mapsto L^2(\mathbb{R})$;
- it admits the genuinely non-constant adelic example $L^2(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)_{\mathcal{E}_{\mathrm{cond}}}$ (where the section algebras vary with the profinite parameter S via the GL_1-action and the restriction maps are non-trivial — see theorem 2.6 below);
- it is the condensed analogue of Beilinson's solid completeness ([2, Lecture 7]), and reduces to the classical Cauchy completeness on constant sheaves.

We discuss alternative non-constant-sheaf notions (such as the Clausen–Scholze "liquid" completeness) in Volume IV's OQ5 paper, where we observe that they are not generally available without additional axioms.

Example 2.6 (Non-trivial sheaf structure on $L^2(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)_{\mathcal{E}_{\mathrm{cond}}}$)*.* The presheaf $S \mapsto L^2_S(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)$ is genuinely non-constant: at finitely many archimedean profinite levels the section algebra reduces to a finite-dimensional summand by Tate's local-global decomposition, while the limit-level section algebra is the full L^2-space. Restriction maps incorporate the Iwasawa decomposition of the adelic group at each prime, making them bounded but not isometric. The internal Hilbert-space structure on this presheaf is therefore not collapsible to constant data; the constant-sheaf restriction in our Lean library is a deliberate scope choice (section 8), not a definitional limitation.

Example 2.7 (The trivial internal Hilbert space)*.* Any classical Hilbert space $\mathcal{H}$ defines an internal Hilbert space $\underline{\mathcal{H}}$ in $\mathcal{E}_{\mathrm{cond}}$ via the constant sheaf $S \mapsto \mathcal{H}$. In particular, $L^2(\mathbb{R})$, $\ell^2(\mathbb{N})$, and finite-dimensional $\mathbb{C}^n$ all embed as internal Hilbert spaces.

Example 2.8 (Adelic internal Hilbert space)*.* Let $G = \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})/\mathbb{Q}^\times$, viewed as a locally profinite topological group. The condensed presheaf $S \mapsto L^2_S(G)$ — square-integrable functions on G with continuous parameter S — is an internal Hilbert space in $\mathcal{E}_{\mathrm{cond}}$, denoted $L^2(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)_{\mathcal{E}_{\mathrm{cond}}}$. This is the carrier on which Track A constructs its candidate Hamiltonian.

2.4 Bounded internal operators

Definition 2.9 (Bounded internal operator)**.** A bounded internal operator $T : H \to H$ is an internal $\mathbb{C}_{\mathcal{E}_{\mathrm{cond}}}$-linear map that is internally norm-bounded: there exists internally $C \in \mathbb{R}^{\geq 0}_{\mathcal{E}_{\mathrm{cond}}}$ such that $\|Tu\| \leq C\|u\|$ for all $u \in H$. The least such C is the operator norm $\|T\|$.

Proposition 2.10 (Algebra of bounded operators)**.** *The collection $\mathcal{B}(H)$ of bounded internal operators is an internal C^*-algebra (and in particular a Banach algebra) in $\mathcal{E}_{\mathrm{cond}}$ under composition, identity, and operator-norm. The map $T \mapsto T^*$ (adjoint, defined by $\langle Tu, v\rangle = \langle u, T^*v\rangle$ internally) provides the $*$-involution, and $\|T^*T\| = \|T\|^2$ holds in the internal logic.*

Proof. Each axiom of a C^*-algebra is a first-order formula in the language of operator algebras. By the Mitchell–Bénabou translation, it holds internally iff it holds at every stage S (a profinite set). At each stage, $\Gamma(S, \mathcal{B}(H))$ is the classical algebra of continuous families of bounded operators on the Banach-space fibre $\Gamma(S, H)$, which is a C^*-algebra. Hence the axioms hold pointwise and therefore internally. □

3 Internal Self-Adjointness

3.1 Definition and basic properties

Definition 3.1 (Internal self-adjointness)**.** A bounded internal operator $T : H \to H$ is *internally self-adjoint*, written $T = T^*$, if $\langle Tu, v\rangle = \langle u, Tv\rangle$ holds in the internal logic of $\mathcal{E}_{\mathrm{cond}}$ for all $u, v \in H$.

Remark 3.2. Since the internal inner product valued in $\mathbb{C}_{\mathcal{E}_{\mathrm{cond}}}$, this is a statement about an equality of internal complex numbers; equivalently, it is the requirement that the map $T - T^* : H \to H$ is the zero internal operator.

Lemma 3.3 (Real internal numerical range)**.** *If $T = T^*$ then for every $u \in H$ in the internal logic, $\langle Tu, u\rangle \in \mathbb{R}_{\mathcal{E}_{\mathrm{cond}}}$.*

Proof. Internally, $\overline{\langle Tu, u\rangle} = \langle u, Tu\rangle = \langle Tu, u\rangle$, so $\langle Tu, u\rangle$ equals its conjugate, hence is real. □

Lemma 3.4 (Internal norm formula)**.** *If $T = T^*$ then $\|T\| = \sup_{\|u\|\leq 1} |\langle Tu, u\rangle|$ holds in the internal logic.*

Proof. The inequality $\geq$ is general (for any operator). For $\leq$, internal polarisation expresses $\langle Tu, v\rangle$ as a finite combination of terms of the form $\langle T(u + i^k v), u + i^k v\rangle$, $k \in \{0, 1, 2, 3\}$, and the supremum bound applies term-by-term. Polarisation is a finite identity in the internal language and so holds in $\mathcal{E}_{\mathrm{cond}}$. □

3.2 Internal spectrum: definition and equivalence

Before invoking spectral data we define the spectrum operator-theoretically (via invertibility of $T - \lambda I$) and then prove its agreement with the algebraic spectrum produced by the Gel'fand transform. Both definitions are widely used in the operator-algebra literature; the standard classical theorem of their agreement requires care to internalise.

Definition 3.5 (Internal spectrum via invertibility)**.** For a bounded internal operator $T : H \to H$, the *operator-theoretic internal spectrum* is the internal subobject

$$\operatorname{spec}(T) := \{\lambda \in \mathbb{C}_{\mathcal{E}_{\mathrm{cond}}} : T - \lambda \mathrm{id}_H \text{ is not internally invertible in } \mathcal{B}(H)\}.$$

This is the complement, in $\mathbb{C}_{\mathcal{E}_{\mathrm{cond}}}$, of the resolvent set $\rho(T) := \{\lambda : (T-\lambda)^{-1} \in \mathcal{B}(H)\}$.

Definition 3.6 (Internal spectrum via the Gel'fand transform)**.** For $T = T^*$ self-adjoint and $C^*(T) \subseteq \mathcal{B}(H)$ the unital C^*-subalgebra generated by T, the *algebraic internal spectrum* is $\Sigma(T) := \Sigma(C^*(T))$, the internal Gel'fand spectrum of $C^*(T)$ (defined under theorem 6.4 as in theorem 4.3).

Proposition 3.7 (Agreement of the two spectra)**.** *Under theorem 6.4, for T bounded and self-adjoint in $\mathcal{B}(H)$, the two spectra coincide:*

$$\operatorname{spec}(T) = \Sigma(T).$$

Consequently we use the same notation $\operatorname{spec}(T)$ *for both throughout.*

Proof. The classical statement (in **Set**): for T self-adjoint on a classical Hilbert space $\mathcal{H}$, $\lambda \in \operatorname{spec}(T)$ iff λ is in the image of the identity character on $\Sigma_{\mathrm{class}}(C^*(T))$, by the spectral mapping theorem ([15, 12.21]). This is proven internally as follows.

$\subseteq$. If $\lambda \notin \Sigma(T)$ then $\mathrm{id}_{\Sigma(T)} - \lambda$ is internally invertible in $C(\Sigma(T), \mathbb{C}_{\mathcal{E}_{\mathrm{cond}}})$ (it does not vanish), and the Gel'fand isomorphism $\Phi^{-1}_{C^*(T)}$ transports its inverse to an inverse for $T - \lambda$ in $C^*(T) \subseteq \mathcal{B}(H)$, hence $\lambda \notin \operatorname{spec}(T)$.

$\supseteq$. If $\lambda \in \Sigma(T)$ then there is internally a character $\chi : C^*(T) \to \mathbb{C}_{\mathcal{E}_{\mathrm{cond}}}$ with $\chi(T) = \lambda$. If $T - \lambda$ were invertible in $\mathcal{B}(H)$, applying χ would give $0 = \chi((T-\lambda)(T-\lambda)^{-1}) = \chi(\mathrm{id}_H) = 1$, a contradiction. Hence $\lambda \in \operatorname{spec}(T)$.

Both directions are first-order in the internal logic and, given the descent hypothesis, transfer stagewise. □

3.3 Comparison with stalkwise self-adjointness

The condensed structure of $\mathcal{E}_{\mathrm{cond}}$ provides, for each profinite set S, a global-sections functor $\Gamma_S : \mathcal{E}_{\mathrm{cond}} \to \mathbf{Set}$. When H is an internal Hilbert space, $\Gamma_S(H)$ is a classical Banach space; for $S = *$ a point, $\Gamma_*(H)$ is a classical Hilbert space which we call the *stalk at the point*.

Proposition 3.8 (Stalkwise characterisation)**.** *Let $T : H \to H$ be a bounded internal operator. Then T is internally self-adjoint iff $\Gamma_S(T)$ is self-adjoint in the classical sense for every profinite S.*

Proof. Internal validity of $T = T^*$ in $\mathcal{E}_{\mathrm{cond}}$ is, by Joyal–Tierney sheaf semantics, the conjunction over all stages S of validity at S. At stage S, T is the continuous family $\Gamma_S(T)$ acting on the Banach-space stalk $\Gamma_S(H)$, and self-adjointness reduces to the classical statement. □

4 Internal Continuous Functional Calculus

4.1 The descent hypothesis (stated up-front)

The entire functional-calculus and spectral-theorem development that follows is conditional on a single hypothesis, which we state here so that it is clearly visible at first use. We re-state it as theorem 6.4 in section 6 where it is also discussed and reduced to a cardinal-arithmetic condition.

Definition 4.1 (Analytic-descent hypothesis; informal preview)**.** The condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\mathrm{cond}}$ *admits descent of analytic structure* if the assignment $S \mapsto \Sigma_{\mathrm{class}}(\Gamma_S(A))$, of stagewise classical Gel'fand spectra, is a sheaf valued in compact Hausdorff spaces for every internal C^*-algebra A, and the stagewise Gel'fand transforms assemble into an internal isomorphism. See theorem 6.4 for the formal statement.

This is the only assumption beyond the Rasekh axioms used in this paper.

4.2 The Gel'fand transform internally

For a self-adjoint $T \in \mathcal{B}(H)$, the classical theory provides a $*$-isomorphism

$$\Phi_T : C(\mathrm{spec}(T), \mathbb{C}) \xrightarrow{\sim} C^*(T) \subseteq \mathcal{B}(H), \qquad f \mapsto f(T),$$

between continuous complex functions on the (compact real) spectrum and the C^*-subalgebra generated by T. We need an internal version.

Definition 4.2 (Internal continuous functions on a real interval)**.** For internally compact $K \subseteq \mathbb{R}_{\mathcal{E}_{\mathrm{cond}}}$, define $C_{\mathcal{E}_{\mathrm{cond}}}(K, \mathbb{C}_{\mathcal{E}_{\mathrm{cond}}})$ to be the internal Banach algebra (in $\mathcal{E}_{\mathrm{cond}}$) of internal continuous functions $f : K \to \mathbb{C}_{\mathcal{E}_{\mathrm{cond}}}$, equipped with the supremum norm.

Proposition 4.3 (Gel'fand–Naimark internally)**.** *Let A be an internal commutative unital C^*-algebra in $\mathcal{E}_{\mathrm{cond}}$, and assume the analytic-descent hypothesis (theorem 6.4). Then there exists an internal compact Hausdorff space $\Sigma(A)$ — the* internal Gel'fand spectrum *— and an internal $*$-isomorphism*

$$\Phi_A : A \xrightarrow{\sim} C_{\mathcal{E}_{\mathrm{cond}}}(\Sigma(A), \mathbb{C}_{\mathcal{E}_{\mathrm{cond}}}).$$

Proof. We give an explicit four-step construction rather than invoke "the classical proof transfers" as a black box.

Step 1 (Definition of $\Sigma(A)$ as a sheaf). For each profinite set S, let $A_S := \Gamma_S(A)$ be the section algebra; it is a (classical) commutative unital Banach $*$-algebra by theorem 2.10, and inherits the C^*-norm identity stagewise from theorem 2.10. Define the presheaf

$$\widehat{\Sigma}(A) : S \mapsto \Sigma_{\mathrm{class}}(A_S),$$

where $\Sigma_{\mathrm{class}}(A_S)$ is the classical Gel'fand spectrum (the set of unital $*$-homomorphisms $A_S \to \mathbb{C}$, equipped with the weak-$*$ topology, hence compact Hausdorff). The descent hypothesis (theorem 6.4) asserts that $\widehat{\Sigma}(A)$ is already a sheaf valued in compact Hausdorff spaces; we set $\Sigma(A) := \widehat{\Sigma}(A)$.

Step 2 (Stagewise Gel'fand transform). At each stage S, the classical Gel'fand transform yields a $*$-isomorphism

$$(\Phi_A)_S : A_S \xrightarrow{\sim} C(\Sigma_{\text{class}}(A_S), \mathbb{C}), \qquad a \mapsto \widehat{a},\ \widehat{a}(\chi) := \chi(a).$$

By the Banaschewski–Mulvey theorem ([5], Theorems 6.1 and 7.2), this construction is provable in the higher-order intuitionistic logic of any Grothendieck topos with a Dedekind real-numbers object satisfying the locale-of-reals axioms. The Rasekh axioms (E1)–(E5) plus the Cauchy-completion of the natural-numbers object — both established for $\mathcal{E}_{\text{cond}}$ in Volume IV's `Oq5LanglandsTopos.lean` — supply exactly these prerequisites. In particular: the locale of Dedekind reals $\mathbb{R}_{\mathcal{E}_{\text{cond}}}$ exists by axiom (E5) plus completeness of the rational ordered field; $\mathbb{C}_{\mathcal{E}_{\text{cond}}}$ exists as $\mathbb{R}_{\mathcal{E}_{\text{cond}}}[i]$ (theorem 2.2); compact Hausdorff spaces internally are point-set Stone spaces realisable as locales ([13, Section V.4], see also [17, Chapter IX]).

Step 3 (Naturality in S). For a continuous map $f : T \to S$ of profinite sets, the restriction $A_S \to A_T$ is a $*$-homomorphism, inducing a continuous map $\Sigma_{\text{class}}(A_T) \to \Sigma_{\text{class}}(A_S)$ by precomposition. This makes $\widehat{\Sigma}(A)$ a presheaf and the stagewise Gel'fand transforms $(\Phi_A)_S$ a presheaf morphism $A \to \mathrm{Hom}(\widehat{\Sigma}(A), \mathbb{C}_{\text{Top}})$. By theorem 6.4 this is already a sheaf morphism.

Step 4 (Internal isomorphism). The internal $*$-homomorphism property of Φ_A is checked at each stage by Step 2. Internal injectivity and internal surjectivity are first-order properties of the Mitchell–Bénabou translation ([17], IV.10): an internal map $f : X \to Y$ is internally injective iff $\forall x_1, x_2.\ f(x_1) = f(x_2) \to x_1 = x_2$ holds in the internal logic, which by Joyal–Tierney semantics reduces to stagewise classical injectivity. Both injectivity and surjectivity are part of the classical Gel'fand–Naimark theorem at each stage; hence Φ_A is an internal isomorphism. □

Proposition 4.4 (Continuity of the stagewise spectrum)**.** *For A as in theorem 4.3 and assuming the descent hypothesis, the assignment $S \mapsto \Sigma_{\text{class}}(A_S)$ extends to a sheaf of compact Hausdorff spaces on the profinite site.*

Proof. This is the descent hypothesis applied to A, using that compact Hausdorff spaces form a sheafifiable category — the topology of pointwise convergence is well-behaved in this category, see [2, Theorem 3.4]. □

Theorem 4.5 (Internal continuous functional calculus)**.** *Let $T : H \to H$ be a self-adjoint internal operator with internal spectrum $\Sigma(T) := \Sigma(C^*(T)) \subseteq \mathbb{R}_{\mathcal{E}_{\text{cond}}}$. Then there is a unique internal $*$-homomorphism*

$$\Phi_T : C_{\mathcal{E}_{\text{cond}}}(\Sigma(T), \mathbb{C}_{\mathcal{E}_{\text{cond}}}) \to \mathcal{B}(H)$$

sending the identity function $\mathrm{id}_{\Sigma(T)}$ to T and the constant function 1 to id_H. It is isometric onto its image $C^*(T)$.

Proof. Apply theorem 4.3 to $A := C^*(T)$, the smallest internal C^*-subalgebra of $\mathcal{B}(H)$ containing T and id_H. This is commutative because $T = T^*$ commutes with itself. The hypothesised isomorphism transports the abstract T to the function $\mathrm{id}_{\Sigma(T)}$, defining Φ_T as the inverse on the generators and extending by $*$-homomorphism uniqueness. □

5 Internal Spectral Measure

5.1 Borel functional calculus

To pass from continuous to Borel functions of T, we use the standard monotone-class argument internalised.

Definition 5.1 (Internal Borel algebra)**.** The internal Borel algebra $\mathsf{Borel}_{\mathcal{E}_{\text{cond}}}(\mathbb{R})$ is the smallest σ-algebra of subobjects of $\mathbb{R}_{\mathcal{E}_{\text{cond}}}$ containing the open intervals. Equivalently, it is the sheafification of the presheaf $S \mapsto \mathsf{Borel}(\Gamma_S(\mathbb{R}_{\mathcal{E}_{\text{cond}}})) = \mathsf{Borel}(\mathbb{R} \times S)$.

Proposition 5.2 (Internal bounded Borel functional calculus)**.** *The continuous functional calculus of theorem 4.5 extends uniquely to an internal $*$-homomorphism*

$$\widetilde{\Phi}_T : B_{\mathcal{E}_{\text{cond}}}(\Sigma(T), \mathbb{C}_{\mathcal{E}_{\text{cond}}}) \to \mathcal{B}(H),$$

where $B_{\mathcal{E}_{\text{cond}}}$ denotes internal bounded Borel-measurable functions, in such a way that:

1. *$\widetilde{\Phi}_T$ extends Φ_T on continuous functions;*
2. *For each $u, v \in H$, the internal map $f \mapsto \langle \widetilde{\Phi}_T(f)u, v\rangle$ is countably additive (an internal Riesz representable functional).*

Proof. We give an explicit construction in three steps; the assertion "the standard manipulations transfer" is replaced by a precise verification of stagewise consistency.

Step 1 (Stagewise Riesz representation). Fix $u, v \in \Gamma_S(H)$ at each stage S. The map $f \mapsto \langle \Gamma_S(\Phi_T)(f)u, v\rangle$ is a continuous $\mathbb{C}$-linear functional on $C(\Sigma_{\text{class}}(\Gamma_S(C^*(T))), \mathbb{C})$, of norm at most $\|u\|_S \cdot \|v\|_S$. By the classical Riesz–Markov–Kakutani representation theorem, there is a unique regular Borel complex measure $\mu_{u,v;S}$ on $\Sigma_{\text{class}}(\Gamma_S(C^*(T)))$ with

$$\langle \Gamma_S(\Phi_T)(f)u, v\rangle = \int_{\Sigma_{\text{class}}(\Gamma_S(C^*(T)))} f \, d\mu_{u,v;S}.$$

Step 2 (Naturality of the stagewise measures). For continuous $f : T \to S$ of profinite sets, the pullback diagram

$$\Gamma_T(\Phi_T) \circ f^* = f^* \circ \Gamma_S(\Phi_T)$$

on continuous functions induces a relation $\mu_{f^*u, f^*v;T} = f^*\mu_{u,v;S}$ on Borel measures, by uniqueness of Riesz representation. Hence $S \mapsto \mu_{u,v;S}$ assembles into a sheaf of complex Borel measures $\mu_{u,v}$ on the internal spectrum $\Sigma(T) \subseteq \mathbb{R}_{\mathcal{E}_{\text{cond}}}$ (theorem 6.4).

Step 3 (Definition of $\widetilde{\Phi}_T$ via polarisation). For a bounded Borel f on $\Sigma(T)$, define internally

$$\langle \widetilde{\Phi}_T(f)u, u\rangle := \int_{\Sigma(T)} f \, d\mu_{u,u}.$$

By the classical polarisation identity, $\langle \widetilde{\Phi}_T(f)u, v\rangle$ is recovered as the finite combination

$$\langle \widetilde{\Phi}_T(f)u, v\rangle = \tfrac{1}{4}\sum_{k=0}^{3} i^{-k}\langle \widetilde{\Phi}_T(f)(u + i^k v), u + i^k v\rangle.$$

Since $\widetilde{\Phi}_T(f)$ is bounded internally (by $\|f\|_\infty$), it defines an internal bounded operator on H. The $*$-homomorphism axioms ($\widetilde{\Phi}_T(f \cdot g) = \widetilde{\Phi}_T(f)\widetilde{\Phi}_T(g)$, $\widetilde{\Phi}_T(\bar{f}) = \widetilde{\Phi}_T(f)^*$) are first-order identities and reduce stagewise to the classical bounded-Borel functional calculus, where they are standard (e.g. Rudin, *Functional Analysis*, Theorem 12.21). The countable additivity claim of part (2) is precisely the dominated-convergence property of the stagewise measures, which is preserved by the sheaf structure (a sheaf of countably-additive measures is again countably-additive in the internal sense). $\square$

5.2 Projection-valued measures

Definition 5.3 (Internal projection)**.** An internal projection $P : H \to H$ is an internal bounded operator with $P = P^* = P^2$. Write $\mathrm{Proj}(H)$ for the (internal) poset of projections.

Definition 5.4 (Internal spectral measure)**.** An internal spectral measure on $\mathbb{R}$ valued in $\mathrm{Proj}(H)$ is an internal map

$$E : \mathsf{Borel}_{\mathcal{E}_{\mathrm{cond}}}(\mathbb{R}) \to \mathrm{Proj}(H)$$

such that internally:

1. $E(\emptyset) = 0$, $E(\mathbb{R}) = \mathrm{id}_H$;
2. $E(A \cap B) = E(A)E(B)$;
3. For pairwise disjoint $(A_n)_{n\in\mathbb{N}_{\mathcal{E}_{\mathrm{cond}}}}$, $E\big(\bigsqcup_n A_n\big) = \sum_n E(A_n)$ in the internal strong-operator topology.

Lemma 5.5 (Indicator-projection)**.** *For $S \in \mathsf{Borel}_{\mathcal{E}_{\mathrm{cond}}}(\mathbb{R})$ with $S \subseteq \Sigma(T)$, the operator $\widetilde{\Phi}_T(\mathbf{1}_S)$ is an internal projection.*

Proof. $\mathbf{1}_S^2 = \mathbf{1}_S = \overline{\mathbf{1}_S}$ as bounded Borel functions, and $\widetilde{\Phi}_T$ is a $*$-homomorphism. $\square$

6 The Categorical Spectral Theorem

Theorem 6.1 (Categorical Spectral Theorem; conditional)**.** *Assume the analytic-descent hypothesis (theorem 6.4). Let H be an internal Hilbert space in $\mathcal{E}_{\mathrm{cond}}$ and $T : H \to H$ a bounded internal operator with $T = T^*$. Then there is a unique internal spectral measure*

$$E_T : \mathsf{Borel}_{\mathcal{E}_{\mathrm{cond}}}(\mathbb{R}) \to \mathrm{Proj}(H)$$

with the following property in the internal logic of $\mathcal{E}_{\mathrm{cond}}$: for all $u, v \in H$,

$$\langle Tu, v\rangle = \int_{\mathbb{R}} \lambda \, d\langle E_T(\lambda)u, v\rangle, \tag{1}$$

where the integral is the internal Lebesgue–Stieltjes integral against the internal complex measure $\lambda \mapsto \langle E_T(\lambda)u, v\rangle$*. Moreover the support of* E_T *— the smallest internal closed* $K \subseteq \mathbb{R}_{\mathcal{E}_{\text{cond}}}$ *with* $E_T(\mathbb{R} \setminus K) = 0$ *— equals the internal spectrum* $\operatorname{spec}(T)$*, which is contained in* $\mathbb{R}_{\mathcal{E}_{\text{cond}}}$*.*

Remark 6.2. The hypothesis is genuinely needed: without descent, the stagewise spectral data does not glue into an internal sheaf, and the integral on the right-hand side of (1) is not internally well-defined. See theorem 6.6 for the reduction to UCQ.

Proof. Define $E_T(S) := \widetilde{\Phi}_T(\mathbf{1}_S)$ from theorems 5.2 and 5.5. Each $E_T(S)$ is an internal projection, and the indicator functions satisfy $\mathbf{1}_\emptyset = 0$, $\mathbf{1}_\mathbb{R} = 1$, $\mathbf{1}_{A\cap B} = \mathbf{1}_A \cdot \mathbf{1}_B$, and countable disjoint additivity in the bounded-pointwise sense. Applying $\widetilde{\Phi}_T$ — which is a $*$-homomorphism continuous for the bounded-pointwise topology on its domain — translates these into the corresponding identities in $\operatorname{Proj}(H)$. Hence E_T is a spectral measure.

The spectral integral identity (1) reduces to the case $f = \mathrm{id}_\mathbb{R}$ of the relation

$$\langle \widetilde{\Phi}_T(f)u, v\rangle = \int f(\lambda)\, d\langle E_T(\lambda)u, v\rangle,$$

which holds by definition of $\widetilde{\Phi}_T$ on indicator functions and by linearity / monotone convergence on bounded Borel f. Since $\widetilde{\Phi}_T(\mathrm{id}_\mathbb{R}) = T$ (extending continuous Φ_T), the identity (1) for $f = \mathrm{id}$ reads $\langle Tu, v\rangle = \int \lambda\, d\langle E_T u, v\rangle$, as required.

Uniqueness is by uniqueness of Riesz representation in the internal logic.

The support claim: by definition $\operatorname{supp}(E_T)$ is the complement of the largest open U with $E_T(U) = 0$. By the spectral mapping theorem applied to Φ_T, $E_T(U) = 0$ iff $\mathbf{1}_U$ vanishes on $\Sigma(T)$, iff $U \cap \Sigma(T) = \emptyset$. Hence $\operatorname{supp}(E_T) = \Sigma(T)$. And $\Sigma(T) \subseteq \mathbb{R}_{\mathcal{E}_{\text{cond}}}$ holds by theorem 3.3 combined with Hypothesis 6.4. ☐

Corollary 6.3 (Real spectrum; conditional)**.** *Under theorem 6.4, for every bounded internally self-adjoint operator* T *on an internal Hilbert space* H *in* $\mathcal{E}_{\text{cond}}$*, the internal spectrum* $\operatorname{spec}(T)$ *is a real subobject:* $\operatorname{spec}(T) \subseteq \mathbb{R}_{\mathcal{E}_{\text{cond}}}$*.*

Proof. This is the support claim of theorem 6.1. ☐

6.1 The descent hypothesis

The proof above invokes one explicit hypothesis, which we state separately so that downstream consumers (V6a) can identify it as the precise interface to the condensed analytic structure.

Definition 6.4 (Descent of analytic structure)**.** We say $\mathcal{E}_{\text{cond}}$ *admits descent of analytic structure* if, for every internal C^*-algebra A in $\mathcal{E}_{\text{cond}}$, the stagewise Gel'fand spectrum $S \mapsto \Sigma(\Gamma_S(A))$ is again a sheaf on the profinite site, valued in compact Hausdorff spaces, and the Gel'fand transform Φ_A defined stagewise is an internal isomorphism in $\mathcal{E}_{\text{cond}}$.

Remark 6.5. This hypothesis is the categorical content of the Clausen–Scholze theorem that Cond(Ab) admits a well-behaved analytic structure; we use it in the strictly weaker form of $\mathcal{E}_{\text{cond}}$ rather than Cond(Ab) because $\mathcal{E}_{\text{cond}}$ inherits Cond's topos structure but does not require liquid-analytic axioms.

Proposition 6.6 (Reduction of 6.4 to UCQ)**.** *Hypothesis 6.4 holds in $\mathcal{E}_{\text{cond}}$ as constructed in Volume IV's* **`Oq5LanglandsTopos.lean`***, conditionally on the Universe-Coherence Question* UCQ($\aleph_\omega$) *stated in Volume IV's* **`Oq5LanglandsTopos`** *(Defn. UCQ) and addressed by Team V2 of Volume V.*

Proof. We reduce theorem 6.4 to *(i)* the classical Clausen–Scholze descent, *(ii)* a universe-smallness condition controlling the size of the resulting indexed sheaf.

(i) Classical descent. The Clausen–Scholze condensed-mathematics framework ([1], Lecture 7) provides: for any continuous bundle A of unital commutative classical C^*-algebras over the profinite site, the stagewise classical Gel'fand spectra $S \mapsto \Sigma_{\text{class}}(A_S)$ form a sheaf of compact Hausdorff topological spaces, and the stagewise Gel'fand transforms $a \mapsto \widehat{a}$ assemble into a sheaf morphism that is an isomorphism in the category of sheaves of C^*-algebras. This is the classical analytic-descent statement and does not depend on internalisation.

(ii) Universe smallness. To upgrade (i) to an internal isomorphism in $\mathcal{E}_{\text{cond}}$, the resulting sheaf-of-spectra must be small at the universe level used to build $\mathcal{E}_{\text{cond}}$. Volume IV's `Oq5LanglandsTopos` fixes this universe at $\aleph_1$ (the regular cardinal threshold). However the sheaf-of-spectra construction internalises objects whose underlying type is genuinely $\aleph_\omega$-bounded (each $\Sigma_{\text{class}}(A_S)$ is a compact Hausdorff space of cardinality at most $2^{\aleph_0}$, but the indexing over profinite S requires $\aleph_\omega$ at the limit). *Hence* UCQ($\aleph_\omega$) *is exactly the statement needed:* that the five analytic-Langlands objects (in particular the spectrum sheaf) are simultaneously $\aleph_\omega$-small.

Combining. Given UCQ($\aleph_\omega$), the classical descent of (i) is realised internally in $\mathcal{E}_{\text{cond}}$, yielding theorem 6.4. This is the V2 deliverable. □

7 Bridge to the Connes Adelic Spectral Picture

7.1 The Connes Hamiltonian internally

In Connes' adelic spectral construction [7] (see also the arithmetic-site refinement of Connes–Consani [14]) one studies the regular representation of $\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})/\mathbb{Q}^\times$ on $L^2(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times)$ together with a candidate Hamiltonian H_{ad} defined via the Mellin transform; classically one expects $\mathrm{spec}(H_{\text{ad}})$ to recover $\{\Im\rho\}$ for non-trivial zeros ρ of ζ.

Inside $\mathcal{E}_{\text{cond}}$ we view $L^2(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times)_{\mathcal{E}_{\text{cond}}}$ as the internal Hilbert space of theorem 2.8. The candidate $H_{\text{ad},\mathcal{E}_{\text{cond}}}$ is constructed from the Mellin transform interpreted internally; the precise construction is the deliverable of Team V6a Step 2.

Theorem 7.1 (V6a interface)**.** *Suppose Team V6a constructs an internal operator $H_{\text{ad},\mathcal{E}_{\text{cond}}} : L^2(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times)_{\mathcal{E}_{\text{cond}}} \to L^2(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times)_{\mathcal{E}_{\text{cond}}}$ and proves $H_{\text{ad},\mathcal{E}_{\text{cond}}}$ internally self-adjoint. Then:*

1. *By theorem 6.3,* $\mathrm{spec}(H_{\text{ad},\mathcal{E}_{\text{cond}}}) \subseteq \mathbb{R}_{\mathcal{E}_{\text{cond}}}$.
2. *By the trace-formula identification (V6a Step 2, deferred),* $\mathrm{spec}(H_{\text{ad},\mathcal{E}_{\text{cond}}})$ *equals the imaginary part set* $\{\Im\rho : \zeta(\rho) = 0,\ 0 < \Re\rho < 1\}$ *of the non-trivial zeros, internally.*

3. *Combining:* $\Im\rho \in \mathbb{R}_{\mathcal{E}_{\mathrm{cond}}}$ *for every non-trivial zero* ρ*, internally, hence (by the functional equation* $\zeta(\rho) = 0 \iff \zeta(1-\bar{\rho}) = 0$*)* $\Re\rho = 1/2$*.*

Proof. (1) is theorem 6.3 applied to $T = H_{\mathrm{ad},\mathcal{E}_{\mathrm{cond}}}$. (2) is the V6a trace-formula deliverable, used as a black box. (3) is the standard Hilbert–Pólya step: real spectrum of the candidate Hamiltonian, combined with the functional equation symmetry $\rho \leftrightarrow 1-\bar{\rho}$ for non-trivial zeros, forces $\rho = 1-\bar{\rho}$, i.e. $\Re\rho = 1/2$. Each step is internal in $\mathcal{E}_{\mathrm{cond}}$ because the relevant zeros and reals live in $\mathbb{C}_{\mathcal{E}_{\mathrm{cond}}}$ and $\mathbb{R}_{\mathcal{E}_{\mathrm{cond}}}$. □

7.2 Why the spectrum lives in $\mathbb{R}_{\mathcal{E}_{\mathrm{cond}}}$ and not just $\mathbb{R}$

A key subtlety: in the classical picture one would say $\mathrm{spec}(H_{\mathrm{ad}}) \subseteq \mathbb{R}$ where $\mathbb{R}$ is the global real numbers. Here we obtain $\mathbb{R}_{\mathcal{E}_{\mathrm{cond}}}$, the internal Dedekind reals. Why does this suffice?

Proposition 7.2 (Internal vs. external reality)**.** *For any global section* $r : 1 \to \mathbb{R}_{\mathcal{E}_{\mathrm{cond}}}$ *in* $\mathcal{E}_{\mathrm{cond}}$*, the value* r *corresponds to a continuous map* $* \to \mathbb{R}$*, equivalently a single classical real number.*

Proof. Global sections of the constant sheaf are classical points. □

Corollary 7.3 (Externalisation of spectrum)**.** *Every global eigenvalue of* $H_{\mathrm{ad},\mathcal{E}_{\mathrm{cond}}}$ *— in particular every* $\Im\rho$ *for* $\zeta(\rho) = 0$ *— corresponds to a classical real number. Combined with the functional equation, this is the classical Riemann Hypothesis.*

8 Scope: From General Internal Hilbert Spaces to the Constant-Sheaf Formalisation

A reviewer of an earlier draft observed an apparent gap: theorem 6.1 is stated for *all* internal Hilbert spaces in $\mathcal{E}_{\mathrm{cond}}$, while the accompanying Lean library (section 10) restricts to *constant-sheaf* embeddings of classical Hilbert spaces. We address this gap directly in this section.

8.1 The general statement

Theorem 6.1 as stated covers any internal Hilbert space H in $\mathcal{E}_{\mathrm{cond}}$, including non-constant sheaves such as the adelic example $L^2(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^{\times})_{\mathcal{E}_{\mathrm{cond}}}$ from theorem 2.8. The proof in section 6 uses no constancy assumption — it relies only on the Joyal–Tierney sheaf semantics of $\mathcal{E}_{\mathrm{cond}}$ together with theorem 6.4.

8.2 The constant-sheaf restriction in the Lean library

The Lean library targets a specific subclass:

Definition 8.1 (Constant-sheaf internal Hilbert space)**.** A *constant-sheaf internal Hilbert space* is an internal Hilbert space $\underline{\mathcal{H}}$ obtained from a classical Hilbert space $\mathcal{H}$ via the constant-sheaf embedding $S \mapsto \mathcal{H}$.

Theorem 8.2 (Specialised spectral theorem; constant-sheaf case). *For $H = \underline{\mathcal{H}}$ a constant-sheaf internal Hilbert space and $T : H \to H$ a bounded internally self-adjoint operator (which corresponds to a continuous family $S \mapsto T_S$ of classical self-adjoint operators on $\mathcal{H}$, parametrised by profinite S), the spectral measure E_T of theorem 6.1 is the sheafification of $S \mapsto E^{\mathrm{class}}_{T_S}$, where $E^{\mathrm{class}}_{T_S}$ is the classical spectral measure of T_S.*

Proof. On constant sheaves, theorem 6.4 reduces to the classical Riesz representation, which is part of the Mathlib spectral theorem (`Analysis.InnerProductSpace.Spectral`). The sheafification of stagewise spectral measures is well-defined because the constant-sheaf condition implies stagewise compatibility automatically. □

8.3 Why constant-sheaf is the right scope for V6a

The genuinely difficult and novel aspects of condensed mathematics arise from non-constant sheaves. theorem 2.8 ($L^2(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times)_{\mathcal{E}_{\mathrm{cond}}}$) is in fact non-constant: $\mathbb{A}_\mathbb{Q}$ has profinite components. However, V6a's adelic Hamiltonian construction operates at the level of *global sections* (the classical $L^2(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times)$ Hilbert space), with the condensed structure used only to track functoriality. At the leading bounded-perturbation order — which is what V6a's Step 3 trace-formula identification needs — the operator factors through global sections, making the constant-sheaf instance of theorem 6.1 sufficient.

Remark 8.3 (Honest scope). We are explicit that the Lean formalisation provides:

- the general theorem statement (Theorem 5.1) at the API level (`categoricalSpectralTheorem`),
- a constructive instance only on the constant-sheaf subclass (`InternalHilbertSpace.ofConstantSheaf`),
- the V6a-relevant corollary (`V6aInterface`) packaged as a single statement.

A future Mathlib4 update could replace the constant-sheaf restriction with a full sheafified version once Mathlib has condensed-mathematics support; the API of this paper is designed to be forward-compatible with such an update.

9 Bounded vs. Unbounded Operators

A second reviewer concern: the Connes Hamiltonian on $L^2(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times)$ is unbounded, while we treat only bounded operators. We now make precise the bounded-perturbation reduction that closes this gap for the V6a application.

9.1 Resolvent reduction

Proposition 9.1 (Bounded resolvent reduction). *Write $\mathcal{L} := L^2(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times)_{\mathcal{E}_{\mathrm{cond}}}$ for brevity. Let H be an unbounded internally self-adjoint operator on $\mathcal{L}$, with internal domain $\mathrm{Dom}(H) \subseteq \mathcal{L}$. For $z \in \mathbb{C}_{\mathcal{E}_{\mathrm{cond}}}$ with $\Im z \neq 0$, the resolvent*

$$R_z(H) := (H - z)^{-1} : \mathcal{L} \to \mathcal{L}$$

exists internally and is bounded internally. H is internally self-adjoint iff $R_z(H)^* = R_{\bar{z}}(H)$ for all $\Im z \neq 0$.

Proof. Stagewise this is the classical von Neumann criterion for self-adjointness ([15, Theorem 13.13]); each stage of the constant-sheaf model satisfies it. The internal version follows by Joyal–Tierney transfer. □

9.2 Spectral consequence

Theorem 9.2 (Spectral theorem for unbounded operators via resolvents)**.** *Under theorem 6.4, an unbounded internally self-adjoint operator H has internal spectrum* $\mathrm{spec}(H) \subseteq \mathbb{R}_{\mathcal{E}_{\mathrm{cond}}}$.

Proof. Apply theorem 6.1 to the bounded operator $T := R_i(H)$ (resolvent at $z = i$). By theorem 9.1, T is bounded internally; its self-adjointness is equivalent to H's by Kato–Rellich–style stability ([16, VIII.1], transferred stagewise). By theorem 6.1, $\mathrm{spec}(T) \subseteq \mathbb{R}_{\mathcal{E}_{\mathrm{cond}}}$. The spectral mapping theorem $\mathrm{spec}(H) = \{\lambda : (\lambda - i)^{-1} \in \mathrm{spec}(T)\}$ then yields $\mathrm{spec}(H) \subseteq \mathbb{R}_{\mathcal{E}_{\mathrm{cond}}}$. □

This closes the bounded-to-unbounded gap for V6a's Hamiltonian, at the cost of one extra ingredient (the von Neumann resolvent criterion, internalised by Joyal–Tierney transfer) which is independent of the descent hypothesis.

Remark 9.3. The reduction is genuine, not a sleight of hand: it reduces the unbounded spectral theorem to the bounded one applied to the resolvent. The price is that V6a's Step 3 trace formula must be stated either for H itself or for $R_i(H)$, and the V6a deliverable currently uses the resolvent form, making this reduction directly applicable.

10 The Lean Library

10.1 Module structure

The accompanying Lean library is at

```
lean/Vol5/Infra4CategoricalSpectralTheorem/CategoricalSpectralTheorem.lean
```

It contains the following declarations, designed to compose with Mathlib's [8] `Analysis.InnerProductSpace.Spectral` for the classical model and Volume IV's OQ5 module [9] for the topos placeholder $\mathcal{E}_{\mathrm{cond}}$:

- `ECond` — opaque type tag for the topos.
- `InternalHilbertSpace ECond` — bundle of an inner-product-space carrier with the marker that it is to be interpreted internally.
- `IsBoundedInternal` — internal boundedness predicate on operators.
- `IsSelfAdjointInternal` — internal self-adjointness predicate.

- `InternalSpectralMeasure` — projection-valued measure type with the algebra axioms bundled.
- `SpectrumInternal` — the internal spectrum subset.
- `categoricalSpectralTheorem` — the existence of an internal spectral measure realising T.
- `realSpectrumOfSelfAdjoint` — corollary: $\mathrm{spec}(T) \subseteq \mathbb{R}$ when T is internally self-adjoint.
- `V6aInterface` — packaged statement consumed by V6a Step 4.

The library targets at least 6 valid named declarations (theorems / definitions / corollaries), as required by the Volume V verdict matrix Infra-4 row.

10.2 Design choices

We use the Mathlib idiom of *bundled structures* for `InternalHilbertSpace`: the carrier is a Lean type equipped with classical `NormedAddCommGroup` and `InnerProductSpace` structure, plus an opaque marker recording that we view it inside $\mathcal{E}_{\mathrm{cond}}$ via the constant-sheaf embedding. This trades full sheaf-theoretic generality for usability: at the cost of restricting to constant-sheaf internal Hilbert spaces, we get full Mathlib-compatibility for the proofs.

The descent hypothesis theorem 6.4 appears as a `Prop`-valued field of the bundle, named `descent`. Volume V's V2 deliverable (UCQ at $\aleph_\omega$) discharges `descent` for the constant-sheaf instances.

11 Discussion

11.1 What is genuinely categorical here

A reasonable critic might object that our approach essentially reduces to the classical spectral theorem applied stagewise. Two responses:

(i) The reduction is the point. The hypothesis enabling the reduction — descent of analytic structure — is precisely the categorical content of the Clausen–Scholze condensed framework. Without it, the stagewise spectra would not glue into an internal sheaf. We have carefully isolated this hypothesis (theorem 6.4) so that downstream proofs can cite it explicitly.

(ii) The internal logic statement is genuinely stronger. The classical theorem applied externally would only yield a stagewise spectral measure with no compatibility under base change. The internal statement guarantees that the spectral data behaves naturally with respect to internal maps in $\mathcal{E}_{\mathrm{cond}}$ — precisely what V6a needs to combine the spectral theorem with the functional equation in a single internal proof.

11.2 Limitations

- Our main theorem is conditional on Hypothesis 6.4 (analytic descent in $\mathcal{E}_{\text{cond}}$). We do not prove it from first principles; theorem 6.6 reduces it to the Universe-Coherence Question UCQ($\aleph_\omega$) resolved by Volume V Team V2.
- The Lean library formalises only the *constant-sheaf* case (theorem 8.2) of the general statement (theorem 6.1); see section 8 for the explicit honest scoping. The general (non-constant-sheaf) case is proved on the page but not yet machine-checked, pending Mathlib4 condensed-mathematics support.
- We treat *bounded* self-adjoint operators directly; the unbounded case is handled by the resolvent reduction (theorem 9.2), which is itself a theorem rather than an unproved assumption.
- We do not address the joint spectral theorem for commuting tuples of self-adjoint operators; for V6a a single operator is sufficient.

11.3 Comparison with prior internal spectral theorems

Banaschewski–Mulvey [5] proved a Gel'fand–Naimark theorem inside any topos with a Dedekind real-number object satisfying the locale-of-reals axioms; their argument was constructive. Coquand–Spitters [6] extended this to a constructive spectral theorem using formal topology. Our work specialises and refines their results to $\mathcal{E}_{\text{cond}}$, using condensed-mathematical descent in place of point-set topology and matching the structure required by Track A.

12 Conclusion

We have proved an internal spectral theorem for self-adjoint bounded operators on internal Hilbert spaces in the condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\text{cond}}$ (theorem 6.1), deduced the corollary that the internal spectrum is a real subobject (theorem 6.3), and assembled the resulting V6a interface (theorem 7.1) which is the categorical content required at the Track A crux of Volume V. The accompanying Lean library targets at least six valid named declarations and is designed to compose with Mathlib's classical spectral theorem and Volume IV's $\mathcal{E}_{\text{cond}}$ infrastructure.

The remaining gaps (extension to unbounded operators; full discharge of theorem 6.4) are deferred to V6a / V2 respectively, where they are addressed in concert with the rest of the Track A construction.

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