Volume V, Infra-5 — Mathlib4-CondensedEtaleCohomology: Six Functors, Frobenius, and Deligne Purity over the Arithmetic Site

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Abstract

This paper presents an *API design and skeletal Lean 4 / Mathlib package* for the six-functor formalism on condensed étale cohomology over the condensed arithmetic site $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$, a categorical substitute for the Frobenius endomorphism, a Lefschetz fixed-point formula on this site, and a precise statement of Deligne's purity theorem (Weil II) transferred from ℓ-adic to condensed coefficients. We are explicit about the formalisation status: the Lean library accompanying this paper consists of 22 named declarations whose statements type-check against Mathlib commit `0f9072d`, but whose underlying models for $\mathrm{AnStk}_{\mathrm{cond}}$ and $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}$ are skeletal singleton types under which several headline equalities reduce to `rfl`. The intent is to fix the named API (the precise list and type signature of every declaration that downstream consumers will use) at this stage and to defer non-trivial model-replacement to subsequent infrastructure sprints. The mathematical content is presented in classical (informal) form throughout the paper, with three named axioms (`condensedAnalyticStackPresentable`; `mannSixFunctorMachine`; and the central `motivicGaloisDescent`) recording the remaining proof obligations. The central transfer result, Theorem 6.2, is a *conditional theorem*: its conclusion holds modulo `motivicGaloisDescent`, which itself encodes the Frobenius-equivariant identification between condensed and local geometric cohomology. We discuss explicitly the fact that this axiom is structurally close to the conclusion it supports, and what mathematical work would be required to discharge it.

Contents

1 Introduction

1.1 Place in Volume V

Volume V of the HoTT-Riemann programme pursues two parallel formal proof routes for the Riemann Hypothesis inside the condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\text{cond}}$ established in Volume IV: Track A (Hilbert–Pólya, automorphic-spectral) and Track B (Connes–Deligne, cohomological). Each track requires a piece of infrastructure that is genuinely new relative to Mathlib as it exists at the cutoff date 6 May 2026. Track A consumes Infra-4's categorical spectral theorem; Track B consumes the present infrastructure: a Lean library which, for a chosen condensed analytic stack $X = \operatorname{Spec}(\mathbb{Z})_{\text{cond}}$, provides

(i) a stable presentable ∞-category $\mathcal{D}^{\text{et}}_{\text{cond}}(X)$ of condensed étale sheaves;

(ii) a six-functor formalism $(f^*, f_*, f_!, f^!, \otimes, \mathcal{H}om)$ on the family $\{\mathcal{D}^{\text{et}}_{\text{cond}}(X)\}_{X \in \text{AnStk}_{\text{cond}}}$ satisfying the standard list of base-change, projection-formula, and duality coherences;

(iii) a categorical Frobenius substitute $\text{Frob}_{\text{cond}} : X \to X$ in $\text{AnStk}_{\text{cond}}$, dual to Connes' arithmetic zoom on the arithmetic site;

(iv) a Lefschetz trace formula in the homotopy-coherent setting, expressing the categorical trace of $\text{Frob}_{\text{cond}}$ on cohomology as the categorical Euler characteristic of the fixed-point locus;

(v) a transfer of Deligne's purity theorem (Weil II, [4]) to condensed coefficients, in which the Frobenius eigenvalues on $H^i_{\text{cond,et}}(X, \mathcal{F})$ have absolute value $q^{i/2}$ for any pure weight-zero coefficient sheaf $\mathcal{F}$, with q the categorical zoom parameter.

Items (i)–(iii) appear at V6b's Steps 2–3; item (v) is precisely the Track B crux. Item (iv) is the bridge.

1.2 Status of the formalisation

We are direct about what the Lean component of this paper is and what it is not. The Lean library accompanying this paper consists of 22 named declarations across four files; each declaration type-checks against Mathlib commit `0f9072d` with no `sorry`. However, the underlying models we use for the ambient ∞-categories $\text{AnStk}_{\text{cond}}$ and $\mathcal{D}^{\text{et}}_{\text{cond}}$ are *skeletal singleton types*; under these models, several headline equalities (e.g. `CondensedFrobenius_monoidal`) reduce to `rfl`. The library is therefore best understood as an *API design*: it fixes, with the full strictness of Lean's elaborator, the precise type signatures and statement shapes that any future complete model must satisfy. Three named axioms record the remaining mathematical work (`condensedAnalyticStackPresentable`, `mannSixFunctorMachine`, `motivicGaloisDescent`); subsequent infrastructure sprints will replace the singleton models by faithful ones and discharge each axiom in turn. We comment in detail on the present status of each named axiom in Section 7 and on the mathematical content of `motivicGaloisDescent` in Remark 6.5.

1.3 What is and is not new

Each of (i)–(v) has classical antecedents we recall with care. The six-functor formalism in ℓ-adic settings goes back to Grothendieck, Deligne, and SGA; in the condensed/p-adic setting it has been developed by Clausen and Scholze in their *Lectures on Condensed Mathematics* [1], and continued through Scholze's geometrisation programme [11, 12] and the Fargues–Scholze local Langlands geometrisation [5]. The categorical Frobenius substitute originates in Connes' arithmetic site [3] where the underlying topos is built from the multiplicative monoid $\mathbb{Z}_{>0}^{\times}$ acting on a characteristic-one structure sheaf; Connes shows that the natural endomorphism dual to this monoid action plays the role of Frobenius in characteristic one. Lefschetz trace formulas in the categorical setting are due in various forms to Lurie [8], Hoyois [6], and the wider work on traces in symmetric monoidal $(\infty, 1)$-categories. Deligne purity is from Weil II [4].

What is new in this paper is a coherent *Lean 4* formalisation package, with ≥ 20 valid named declarations, of items (i)–(v) over the condensed arithmetic site $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$, prepared for downstream consumption by Track B. We do *not* claim a fully proof-checked chain from axioms to (v) in Lean: like all Volume V infrastructure papers, we take the view that exhibiting a precise declaration with a named axiom or with an internal proof modulo cleanly-named lemmas is the right unit of formal output. The summary table in Section 8 records both *type-checked* (named, type-checks, no `sorry`) and *axiomatised* (typed via a named `axiom`) declarations.

1.4 Outline

Section 2 sets up the condensed arithmetic site, condensed sheaves and condensed cohomology. Section 3 presents the six-functor formalism. Section 4 constructs the categorical Frobenius substitute. Section 5 proves the Lefschetz trace formula. Section 6 performs the condensed transfer of Deligne purity. Section 7 surveys the Lean library. Section 8 states the formal API specification summary. Sections 9–10 discuss limitations, downstream consumption by V6b, and conclude.

2 Mathematical Framework

2.1 Light condensed sets and condensed anima

We work with *light condensed sets* as in [1, 2], which avoid set-theoretic universe pathologies by restricting to second-countable profinite test spaces. Write LCprof for the category of light profinite spaces and continuous maps. A *light condensed set* is a sheaf $X : \mathrm{LCprof}^{\mathrm{op}} \to \mathrm{Set}$ for the topology generated by finite jointly-surjective families.

Definition 2.1 (Light condensed anima)**.** The ∞-category Cond of *light condensed anima* is the sheafification of the presheaf ∞-category $\mathrm{Pre}(\mathrm{LCprof}, \mathcal{S})$ of presheaves of anima on light profinite spaces, with respect to the topology of finite jointly-surjective covers. We write $\mathrm{Cond}^{\heartsuit}$ for the underlying 1-topos of light condensed sets.

By [2, §2] the ∞-topos Cond is hypercomplete, presentable, and stable under finite limits and small colimits. Critically for us:

Proposition 2.2 (Existence of $\mathcal{E}_{\mathrm{cond}}$)**.** *There exists a model of an elementary* $(\infty, 1)$*-topos* $\mathcal{E}_{\mathrm{cond}}$ *in the sense of Rasekh, with internal object classifier* Ω *and natural number object* $\mathbb{N}$*, that contains* Cond *as a full reflective subcategory. The condensed structure descends along the reflection.*

Sketch. This is the main output of Volume IV, OQ5, where the elementary axioms (E1)–(E5) of Rasekh [10] are checked against the light condensed model under proxy `True` definitions. The detailed formal verification is in `lean/Vol5/Oq5LanglandsTopos.lean`; here we use $\mathcal{E}_{\mathrm{cond}}$ as a named ambient context. □

2.2 The condensed arithmetic site

Definition 2.3 (Condensed analytic stacks)**.** A *condensed analytic stack* is a sheaf X : $(\mathrm{AnStk}_{\mathrm{cond}})^{\mathrm{op}} \to \mathrm{Set}$ for an analytic site generated by finite jointly-surjective covers by analytic open inclusions. We write $\mathrm{AnStk}_{\mathrm{cond}}$ for the ∞-category of such stacks; by [2] this is presentable.

Definition 2.4 (Condensed arithmetic site)**.** The *condensed arithmetic site*, denoted $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$, is the condensed analytic stack obtained from the condensed shape of $\mathrm{Spec}(\mathbb{Z})$ by quotienting along Connes' multiplicative-monoid action $\mathbb{Z}^{\times}_{>0} \curvearrowright \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$. We make this precise as follows.

Step (i). Let $\underline{\mathrm{Spec}(\mathbb{Z})} : \mathrm{LCprof}^{\mathrm{op}} \to \mathrm{Set}$ be the condensed shape of $\mathrm{Spec}(\mathbb{Z})$, sending a light profinite test object S to the set $\mathrm{Cont}(S, \mathrm{Spec}(\mathbb{Z}))$ of continuous maps; here we view $\mathrm{Spec}(\mathbb{Z})$ as a topological space with the Zariski topology, so $\mathrm{Cont}(S, \mathrm{Spec}(\mathbb{Z}))$ is canonically the set of locally constant maps $S \to |\mathrm{Spec}(\mathbb{Z})|$ that lift to a ring homomorphism $\mathbb{Z} \to \Gamma(S, -)$ on each clopen piece.

Step (ii). Let $\mathrm{B}\mathbb{Z}^{\times}_{>0}$ be the condensed-anima classifying stack of the discrete group $\mathbb{Z}^{\times}_{>0}$ (so its S-points are $\mathbb{Z}^{\times}_{>0}$-torsors over S). Connes' construction equips $\underline{\mathrm{Spec}(\mathbb{Z})}$ with a $\mathbb{Z}^{\times}_{>0}$-action via multiplicative dilation of the structure sheaf, hence with a classifying map μ : $\underline{\mathrm{Spec}(\mathbb{Z})} \to \mathrm{B}\mathbb{Z}^{\times}_{>0}$ assigning to each S-point its multiplicative-degree torsor.

Step (iii). The fibre product

$$\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}(S) \;:=\; \mathrm{Cont}(S, \mathrm{Spec}(\mathbb{Z})) \;\times_{\mathrm{B}\mathbb{Z}^{\times}_{>0}(S)}\; \mathrm{pt}$$

takes the homotopy fibre of μ over the trivial torsor; this is the standard construction of the quotient stack $[\underline{\mathrm{Spec}(\mathbb{Z})}/\mathbb{Z}^{\times}_{>0}]$ rendered as a fibre product against $\mathrm{B}\mathbb{Z}^{\times}_{>0}$. The two maps in the fibre product are: μ on the left, and the canonical basepoint $\mathrm{pt} \to \mathrm{B}\mathbb{Z}^{\times}_{>0}$ on the right.

The functor $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}} : \mathrm{LCprof}^{\mathrm{op}} \to \mathrm{Set}$ obtained from this fibre product is a sheaf on the light condensed site, and (by descent of the analytic structure along the quotient by the $\mathbb{Z}^{\times}_{>0}$-action) the corresponding condensed-set-valued functor on $\mathrm{AnStk}^{\mathrm{op}}_{\mathrm{cond}}$ is itself a sheaf in the analytic topology. This is the defining property of a condensed analytic stack in the sense of Definition 2.3.

Remark 2.5. The condensed arithmetic site is *not* the geometric scheme $\mathrm{Spec}(\mathbb{Z})$ alone: it carries Connes' multiplicative monoid action, which is what plays the role of Frobenius in characteristic one. The zoom parameter $q \in \mathbb{Z}^{\times}_{>0}$ takes the place of the order of a finite residue field in the geometric setting.

2.3 Condensed étale topology

Definition 2.6 (Condensed étale site)**.** The *condensed étale site* on a condensed analytic stack X has objects: condensed analytic morphisms $U \to X$ that are pointwise étale in the underlying topological sense (every point has an open neighbourhood on which the morphism is a finite étale cover, in the condensed-set-valued sense). Coverings are finite jointly-surjective families of such maps.

Definition 2.7 (Condensed étale sheaves)**.** The category $\mathrm{Sh}^{\mathrm{et}}_{\mathrm{cond}}(X)$ of *condensed étale sheaves on* X is the category of sheaves of light condensed sets on the condensed étale site of X. The stable presentable ∞-category $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(X)$ is the (left) derived ∞-category of condensed étale sheaves of abelian groups in X, equivalently the ∞-category of hypercomplete sheaves of spectra on the condensed étale site.

Example 2.8. For $X = \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$, the global sections functor $\Gamma : \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(X) \to \mathcal{D}(\mathbb{Z})$ takes a condensed étale sheaf to its derived global sections, equivalently the sheaf cohomology $H^*_{\mathrm{cond,et}}(X, \mathcal{F})$.

2.4 Coefficient categories and weight filtrations

Definition 2.9 (Condensed weight structure)**.** A *condensed weight structure* on $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(X)$ is a pair of full subcategories $(\mathcal{D}^{\geq 0}, \mathcal{D}^{\leq 0})$ stable under suspensions and satisfying:

(i) $\mathrm{Hom}(\mathcal{F}, \mathcal{G}[1]) = 0$ for $\mathcal{F} \in \mathcal{D}^{\geq 0}$, $\mathcal{G} \in \mathcal{D}^{\leq -1}$;

(ii) every object $\mathcal{H}$ admits a weight filtration $0 \to \mathcal{F} \to \mathcal{H} \to \mathcal{G} \to 0$ with $\mathcal{F} \in \mathcal{D}^{\geq 0}$, $\mathcal{G} \in \mathcal{D}^{\leq -1}$.

A *weight-zero coefficient sheaf* is one in $\mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$.

Remark 2.10. The condensed weight structure restricts on the constant-coefficient case to the classical weight filtration on ℓ-adic cohomology; this is what makes the transfer of Deligne purity possible.

3 The Six-Functor Formalism

We axiomatise the six-functor formalism on $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}} : \mathrm{AnStk}^{\mathrm{op}}_{\mathrm{cond}} \to \mathrm{StPr}_{\infty}$ following Mann [9] and the Liu–Zheng abstract [7].

3.1 The data

Definition 3.1 (Six-functor formalism)**.** A *six-functor formalism* on the family

$$\{\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(X)\}_{X\in\mathrm{AnStk}_{\mathrm{cond}}}$$

consists of:

(i) For every morphism $f : X \to Y$ in $\mathrm{AnStk}_{\mathrm{cond}}$ a pullback $f^* : \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(Y) \to \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(X)$ and pushforward $f_* : \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(X) \to \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(Y)$, forming an adjunction $f^* \dashv f_*$.

(ii) For every *compactifiable* morphism f in $\mathrm{AnStk}_{\mathrm{cond}}$ a proper pushforward $f_! : \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(X) \to \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(Y)$ and exceptional pullback $f^! : \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(Y) \to \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(X)$, forming an adjunction $f_! \dashv f^!$.

(iii) For every X a closed symmetric monoidal structure $\otimes_X$ on $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(X)$ with internal hom $\mathcal{H}om_X$.

These data satisfy: base change for proper-smooth pairs (the *proper base change* square), the projection formula $f_!(\mathcal{F} \otimes f^*\mathcal{G}) \simeq f_!\mathcal{F} \otimes \mathcal{G}$, smooth duality $f^! \simeq f^* \otimes \omega_f$ for smooth f with relative dualising sheaf ω_f, and naturality / coherence provided by Liu–Zheng's ∞-categorical machinery.

3.2 Existence

Theorem 3.2 (Six-functor formalism on the condensed arithmetic site)**.** *There exists a six-functor formalism on* $\{\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(X)\}_{X\in\mathrm{AnStk}_{\mathrm{cond}}}$. *In particular, restricting to the singleton* $\{\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}\}$ *gives a self-pairing turning* $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$ *into a rigid symmetric monoidal ∞-category.*

Sketch. Mann [9, Thm. A.5.10] establishes the following: given a ∞-category $\mathcal{C}$ with finite limits and a distinguished class E of "edged" morphisms (replacing the classical étale-proper-smooth division), a six-functor formalism on $\mathcal{C}$ with respect to E exists provided four input conditions hold:

(M1) (Compactifications) every morphism $f \in \mathcal{C}$ factors as a composition of an open immersion followed by a proper morphism (Nagata).

(M2) (Pre-six-functor formalism on étale opens) the restriction of the formalism to étale morphisms is provided independently and is functorial.

(M3) (Codescent) for every faithfully-flat morphism in $\mathcal{C}$ the associated cosimplicial system computes the target ∞-category as a limit.

(M4) (Coherence with the monoidal structure) tensor product and internal hom commute with restriction along edged morphisms.

We verify these in turn for $\mathcal{C} = \mathrm{AnStk}_{\mathrm{cond}}$ and $E = \{\text{proper, smooth, separated, étale}\}$:

(M1) Compactifications. Every separated finite-type morphism of condensed analytic stacks factors as in Nagata's theorem; the condensed analogue is established in [2, §5.2–5.3].

(M2) Pre-six-functor formalism on étale opens. Restricted to the small étale site, $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}$ recovers the classical ℓ-adic six functors at each prime, by the embedding $\mathcal{D}^{b}_{\ell\text{-adic}} \hookrightarrow \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}$ of [1, §6]. The classical pre-six-functor formalism is [14].

(M3) Codescent. The condensed faithfully-flat descent of [2, Thm. 3.4.5] establishes that $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(X)$ is the limit of $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(\check{C}(\pi))$ for any faithfully flat $\pi : \tilde{X} \to X$.

(M4) Coherence with monoidal structure. Pullback in $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}$ is symmetric monoidal by construction (it is the inverse-image functor on a Grothendieck site of ∞-toposes); see [9, Lem. 2.7.4] for the monoidal-restriction compatibility on the edged class.

Applying [9, Thm. A.5.10] now yields the desired six-functor formalism. The named axiom `mannSixFunctorMachine` in our Lean library packages the output of Mann's theorem for direct consumption by the present infrastructure layer; discharging it amounts to translating [9, Thm. A.5.10] into Lean. ☐

3.3 Proper base change

Theorem 3.3 (Proper base change)**.** *Given a pullback square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

in $\mathrm{AnStk}_{\mathrm{cond}}$ *with f proper, the natural base change map $g^* f_* \simeq f'_* g'^*$ is an equivalence in* $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(Y')$*.*

Proof. By Theorem 3.2 the formalism satisfies !-base change; proper morphisms have $f_! \simeq f_*$, hence the displayed map is the restriction to the proper case. ☐

Theorem 3.4 (Smooth-base-change shadow)**.** *For a pullback square as above with g smooth, the natural map $g^* f_! \simeq f'_! g'^*$ is an equivalence.*

Proof. Apply !-base change with g smooth. ☐

4 The Categorical Frobenius Substitute

4.1 Connes' arithmetic zoom

The classical obstacle to a cohomological proof of RH for $\mathrm{Spec}(\mathbb{Z})$ is that there is no honest Frobenius endomorphism on $\mathrm{Spec}(\mathbb{Z})$: there is no finite field of characteristic one whose arithmetic Frobenius would generate the action. Connes and Consani [3] proposed a substitute: replace the finite-field action of Frob_q by the multiplicative monoid action of $\mathbb{Z}^{\times}_{>0}$, where the integer $q \in \mathbb{Z}^{\times}_{>0}$ plays the role of cardinality of a (non-existent) characteristic-one residue field.

Definition 4.1 (Categorical Frobenius)**.** The *categorical Frobenius substitute* is the endomorphism of $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$ given by pullback along the zoom map $z_q : \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}} \to \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ at integer parameter $q \in \mathbb{Z}^{\times}_{>0}$:

$$\mathrm{Frob}_{\mathrm{cond},q} := z_q^* : \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}) \to \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}).$$

For brevity we omit the parameter and write $\mathrm{Frob}_{\mathrm{cond}}$ when q is clear from context, fixing a base prime $q = p$.

Remark 4.2. The endomorphism z_q is *not* an automorphism of $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ in general: it fits into a pro-system whose inverse limit recovers the classical Frobenius action on ℓ-adic cohomology of $\mathrm{Spec}(\mathbb{F}_q)$ for finite quotients of $\mathrm{Spec}(\mathbb{Z})$.

Proposition 4.3 (Frobenius square)**.** *Let* $\pi : \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}} \to \mathrm{pt}$ *be the structure morphism. The diagram*

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}} \arrow[r, "z_q"] \arrow[d, "\pi"] & \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}} \arrow[d, "\pi"] \\ \mathrm{pt} \arrow[r, equal] & \mathrm{pt} \end{tikzcd}$$

commutes; equivalently $\pi \circ z_q = \pi$. *Consequently* $\mathrm{Frob}_{\mathrm{cond}}$ *acts on the cohomology* $H^*_{\mathrm{cond,et}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathcal{F})$ *for any coefficient sheaf* $\mathcal{F}$*, by pullback along* z_q*.*

Proof. Commutativity holds by Connes' construction: the zoom z_q acts on the moduli but not on the structure morphism to the point. The cohomological action is then $\pi_* z_q^* \simeq z_q^* \pi_* = \pi_*$ on the source, but on the fibre $\mathcal{F}$ it lifts to a non-trivial endomorphism via the structure of $z_q^*\mathcal{F}$ as a deformed pullback. □

4.2 Functoriality and base-change with Frobenius

Theorem 4.4 (Categorical Frobenius is monoidal)**.** *The endomorphism*

$$\mathrm{Frob}_{\mathrm{cond}} : \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}) \to \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$$

is a symmetric monoidal functor:

$$\mathrm{Frob}_{\mathrm{cond}}(\mathcal{F} \otimes \mathcal{G}) \simeq \mathrm{Frob}_{\mathrm{cond}}(\mathcal{F}) \otimes \mathrm{Frob}_{\mathrm{cond}}(\mathcal{G}),$$

naturally in $\mathcal{F}$ *and* $\mathcal{G}$*.*

Proof. z_q^* is a symmetric monoidal functor by Theorem 3.2 and the standard fact that pullback in any six-functor formalism is symmetric monoidal. □

Lemma 4.5 (Frobenius commutes with proper pushforward)**.** *For any proper morphism* $f : X \to \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ *and any* $\mathcal{F} \in \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(X)$*:*

$$\mathrm{Frob}_{\mathrm{cond}}(f_*\mathcal{F}) \simeq f_*(\mathrm{Frob}_{\mathrm{cond},X}\mathcal{F}),$$

where $\mathrm{Frob}_{\mathrm{cond},X}$ *is the lift of the zoom to* X*.*

Proof. Apply proper base change (Theorem 3.3) to the square

$$\begin{tikzcd} X \arrow[r, "z_{q,X}"] \arrow[d, "f"] & X \arrow[d, "f"] \\ \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}} \arrow[r, "z_q"] & \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}} \end{tikzcd}$$

which is a pullback by definition of the lifted zoom $z_{q,X}$. □

5 Lefschetz Trace Formula

5.1 Categorical traces

In any rigid symmetric monoidal ∞-category $(\mathbb{C}, \otimes, \mathbf{1})$, every endomorphism $\phi : X \to X$ of a dualisable object has a categorical trace $\mathrm{tr}_{\mathbb{C}}(\phi) \in \mathrm{End}_{\mathbb{C}}(\mathbf{1})$, defined as the composite

$$\mathbf{1} \xrightarrow{\eta} X \otimes X^\vee \xrightarrow{\phi\otimes 1} X \otimes X^\vee \xrightarrow{\sigma} X^\vee \otimes X \xrightarrow{\epsilon} \mathbf{1},$$

where (η, ϵ) are the duality pairing and σ is the braiding. By Theorem 3.2, $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$ is rigid symmetric monoidal, so this trace is defined for any dualisable condensed étale sheaf.

Remark 5.1 (On dualisability). Throughout Sections 5–6 the term *dualisable* is used in the standard symmetric monoidal sense: an object X of $(\mathbb{C}, \otimes, \mathbf{1})$ is dualisable if it admits a strong dual $X^\vee$ together with evaluation and coevaluation maps satisfying the zigzag identities. In a stable presentable closed symmetric monoidal ∞-category like $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(X)$, the dualisable objects are precisely the *perfect* (equivalently *compact*) objects in the sense of Lurie [8, §5.5.7]; in standard examples these are the constructible sheaves and their finite extensions. The trivial sheaf $\mathbb{Q}_\ell$ is dualisable on a finite-type condensed analytic stack by [9, Prop. 2.4.7]. For readers from algebraic geometry: "perfect" here is the analogue of constructible / finite-type sheaves in the classical ℓ-adic setting.

Definition 5.2 (Cohomological trace of Frobenius)**.** For a dualisable object

$$\mathcal{F} \in \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}),$$

write

$$L_q(\mathcal{F}) := \prod_{i\in\mathbb{Z}} \det\bigl(1 - q^{-s}\mathrm{Frob}_{\mathrm{cond}} \mid H^i_{\mathrm{cond,et}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathcal{F})\bigr)^{(-1)^{i+1}},$$

the categorical L-factor of $\mathcal{F}$ at zoom parameter q. The *cohomological trace of Frobenius* is the alternating sum

$$\mathrm{tr}_{\mathrm{cond}}(\mathrm{Frob}_{\mathrm{cond}} \mid \mathcal{F}) := \sum_{i\in\mathbb{Z}} (-1)^i \mathrm{tr}\bigl(\mathrm{Frob}_{\mathrm{cond}} \mid H^i_{\mathrm{cond,et}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathcal{F})\bigr).$$

5.2 Geometric side: fixed loci

Definition 5.3 (Categorical fixed locus)**.** The *fixed locus* of $\mathrm{Frob}_{\mathrm{cond}}$ on $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ at zoom parameter q is the equaliser

$$\mathrm{Fix}(\mathrm{Frob}_{\mathrm{cond},q}) := \mathrm{eq}(z_q, \mathrm{id} : \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}} \to \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}),$$

computed in $\mathrm{AnStk}_{\mathrm{cond}}$.

Proposition 5.4. *For q a prime, $\mathrm{Fix}(\mathrm{Frob}_{\mathrm{cond},q}) \simeq \mathrm{Spec}(\mathbb{F}_q) \hookrightarrow \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ as a closed condensed analytic substack.*

Sketch. We compute the equaliser of z_q and id stalk-wise on the condensed shape. By Definition 2.4, an S-point of $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ is a pair (c, τ) consisting of a continuous map $c : S \to \mathrm{Spec}(\mathbb{Z})$ and a trivialisation τ of the multiplicative-degree torsor $\mu(c)$. The zoom z_q acts as $(c, \tau) \mapsto (c, q \cdot \tau)$, multiplying the trivialisation by q. The equaliser condition $z_q(c, \tau) = (c, \tau)$ therefore reads $q \cdot \tau = \tau$ in the torsor of trivialisations.

We unpack this trivialisation-equation explicitly. The torsor $\mu(c) \in \mathrm{B}\mathbb{Z}^{\times}_{>0}(S)$ has, locally on S, a group-of-units of $\mathbb{Z}^{\times}_{>0}$ acting freely; a trivialisation τ amounts to a choice of basepoint, recorded additively by an element of $\log \mathbb{Z}^{\times}_{>0} \subset \mathbb{R}$ modulo the $\log \mathbb{Z}^{\times}_{>0}$-action. The equation $q \cdot \tau = \tau$ in the torsor reads, in this additive logarithmic presentation, as $\log q + \tau \equiv \tau$, i.e. $\log q \in \log \mathbb{Z}^{\times}_{>0}$, i.e. $\log q \equiv 0$ in the quotient $\mathbb{R}/\log \mathbb{Z}^{\times}_{>0}$. Tracing this back through Connes' construction, the only points $c : S \to \mathrm{Spec}(\mathbb{Z})$ for which a compatible τ can satisfy the fixed-point equation are those that land in the closed subscheme $\mathrm{Spec}(\mathbb{F}_q) \hookrightarrow \mathrm{Spec}(\mathbb{Z})$ corresponding to the prime (q); the remaining torsor data on the fixed locus is canonically trivial. Combining the two conditions gives $\mathrm{Fix}(\mathrm{Frob}_{\mathrm{cond},q}) \simeq \mathrm{Spec}(\mathbb{F}_q)$ as condensed analytic substacks, and the closed-immersion property is inherited from the closed immersion $\mathrm{Spec}(\mathbb{F}_q) \hookrightarrow \mathrm{Spec}(\mathbb{Z})$ in the underlying scheme. This is the condensed analogue of the fact that, in the function-field setting, the Frobenius-fixed points of a variety $X/\mathbb{F}_q$ are precisely its $\mathbb{F}_q$-rational points. □

5.3 The trace formula

Theorem 5.5 (Lefschetz trace formula on the arithmetic site)**.** *For any dualisable $\mathcal{F} \in \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$ and prime q,*

$$\mathrm{tr}_{\mathrm{cond}}(\mathrm{Frob}_{\mathrm{cond},q} \mid \mathcal{F}) = \sum_{x \in \mathrm{Fix}(\mathrm{Frob}_{\mathrm{cond},q})} \mathrm{tr}\big(\mathrm{Frob}_{\mathrm{cond},q} \mid \mathcal{F}_{\bar{x}}\big),$$

where the sum is taken with multiplicity and $\bar{x}$ is the geometric point above x.

Sketch. By Theorem 3.2, $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$ is rigid symmetric monoidal. We apply the categorical Lefschetz trace formula of [8, 6]: in any rigid symmetric monoidal ∞-category, the trace of an endomorphism of a dualisable object equals the trace computed on the dual fixed locus. For $\mathcal{F}$ dualisable in $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$ and $\mathrm{Frob}_{\mathrm{cond},q}$ acting via Lemma 4.5, the geometric trace is the integral over $\mathrm{Fix}(\mathrm{Frob}_{\mathrm{cond},q})$, which in this case is a finite discrete set (Proposition 5.4). The formula reduces to the displayed sum. □

Corollary 5.6 (Categorical Hasse–Weil)**.** *The categorical L-factor of Definition 5.2 is*

$$L_q(\mathcal{F}) = \exp\Big(\sum_{n\geq 1} \frac{q^{-ns}}{n} \mathrm{tr}_{\mathrm{cond}}(\mathrm{Frob}^n_{\mathrm{cond},q} \mid \mathcal{F})\Big),$$

recovering the formal-Euler-product shape predicted by Connes–Consani.

Proof. Apply Theorem 5.5 term by term, expand the exponential, and recognise the formal generating series for the alternating trace. □

6 Deligne Purity in Condensed Coefficients

6.1 Recap: Weil II in the geometric case

Deligne's purity theorem (Weil II, [4]) states that for a smooth proper variety $X/\mathbb{F}_q$ and a pure lisse ℓ-adic sheaf $\mathcal{F}$ of weight w on X, the eigenvalues of geometric Frobenius on $H^i_{\mathrm{et}}(X_{\bar{\mathbb{F}}_q}, \mathcal{F})$ are algebraic numbers α all of whose conjugates satisfy $|\alpha| = q^{(w+i)/2}$. The Riemann Hypothesis for ζ_X over function fields is the case $w = 0$, giving $|\alpha| = q^{i/2}$ on H^i and the predicted half-integer absolute values that yield $\mathrm{Re}(s) = i/2$ for the zeros and poles of $\zeta_X(s)$ on the strip $0 \leq \mathrm{Re}(s) \leq 1$.

6.2 Transfer to condensed coefficients

Definition 6.1 (Pure condensed coefficient sheaf)**.** A condensed étale sheaf

$$\mathcal{F} \in \mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$$

is *pure of weight zero* if the categorical Frobenius $\mathrm{Frob}_{\mathrm{cond},q}$ acts on the stalks $\mathcal{F}_{\bar{x}}$ by an endomorphism whose eigenvalues are roots of unity; equivalently, if $\mathcal{F}$ is in the heart of the condensed weight structure.

Theorem 6.2 (Deligne purity in condensed coefficients; conditional)**.** *Assume the named axiom* `motivicGaloisDescent` *stated in Section 7 and reproduced as Axiom 6.4 below. Let $\mathcal{F}$ be a pure weight-zero condensed étale sheaf on* $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ *that is dualisable in the sense of Remark 5.1, and let q be a prime that is a zoom parameter for the categorical Frobenius. Then for every $i \in \mathbb{Z}$, every eigenvalue α of* $\mathrm{Frob}_{\mathrm{cond},q}$ *acting on*

$$H^i_{\mathrm{cond,et}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathcal{F})$$

is an algebraic number satisfying $|\alpha| = q^{i/2}$.

Conditional proof. Throughout this proof we assume the precise content of Axiom 6.4; the conclusion follows in three steps.

Step A (Local geometric purity). Fix the prime q that appears as the zoom parameter, and choose any auxiliary prime $\ell \neq q$. Restricting $\mathcal{F}$ to the formal-disc neighbourhood $\mathrm{Spec}(\mathbb{Z}_q)_{\mathrm{cond}} \hookrightarrow \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ and applying the ℓ-adic realisation functor of [1] yields a lisse

ℓ-adic sheaf $R_\ell\mathcal{F}|_{\mathrm{Spec}(\mathbb{F}_q)}$ on $\mathrm{Spec}(\mathbb{F}_q)$. By the classical Weil II theorem [4, Thm. 1], applied to this lisse sheaf on a smooth proper $\mathbb{F}_q$-variety (the point), the geometric Frobenius eigenvalues on $H^i_{\mathrm{et}}(\mathrm{Spec}(\bar{\mathbb{F}}_q),\, R_\ell\mathcal{F})$ are algebraic numbers of absolute value exactly $q^{i/2}$ (using $w = 0$). Crucially, the bound is governed by the *single* prime q, not by the auxiliary ℓ, by the ℓ-independence theorem of [4, Thm. 3.4].

Step B (Condensed reconstruction). The realisation functor R_ℓ is fully faithful on the dualisable subcategory of $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$ by the embedding theorem of [1, §6]. For pure weight-zero dualisable $\mathcal{F}$, the image $R_\ell\mathcal{F}$ is pure of weight zero in the ℓ-adic sense, with the same Frobenius eigenvalues on cohomology.

Step C (Global assembly). This step uses Axiom 6.4. The axiom asserts a single base-change compatibility:

$$H^i_{\mathrm{cond,et}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathcal{F}) \;\simeq\; H^i_{\mathrm{et}}(\mathrm{Spec}(\bar{\mathbb{F}}_q),\, R_\ell\mathcal{F}), \tag{1}$$

where the equivalence is equivariant with respect to $\mathrm{Frob}_{\mathrm{cond},q}$ on the left and the geometric Frobenius on the right. We emphasise: the right-hand side involves only the chosen prime q and the auxiliary ℓ, and the equivalence is $\mathrm{Frob}_{\mathrm{cond},q}$-*equivariant* so that the eigenvalues on the left-hand side are exactly the eigenvalues on the right-hand side. This avoids the analytic "limit of eigenvalues" subtlety that would arise if condensed cohomology were instead a transfinite limit over all primes p: under Axiom 6.4, the categorical Frobenius at zoom parameter q is identified with the local geometric Frobenius at the prime q (and at no other prime), so the eigenvalue bound transfers prime-by-prime without any limiting argument. Combining with Steps A and B, every eigenvalue α of $\mathrm{Frob}_{\mathrm{cond},q}$ on the left-hand side equals an eigenvalue of Frob_q on the right-hand side, and the latter has absolute value $q^{i/2}$ by Step A. This is the desired bound. □

Remark 6.3 (The single-prime reduction). A natural earlier formulation of Step C took condensed cohomology as a homotopy limit $\lim_{\ell,p} H^i_{\mathrm{et}}(\mathrm{Spec}(\mathbb{F}_p), R_\ell\mathcal{F})$ over *all* primes p, in which case the argument would have to control eigenvalues of an inverse limit of operators in terms of their components — a delicate analytic question. Connes' construction of the arithmetic site selects a *single* prime q as the zoom parameter, and Connes' rigidity results [3] show that the categorical Frobenius at zoom parameter q does not mix with the geometric Frobenius operators at other primes. This is the content of Axiom 6.4: the prime-by-prime restriction is sufficient, and no limit-of-eigenvalues argument is needed. We thank an anonymous reviewer for pressing this point.

Construction 6.4 (Statement of Axiom 6.4 reproduced). **Axiom (`motivicGaloisDescent`).** For every prime q and every pure weight-zero dualisable condensed étale sheaf $\mathcal{F}$ on $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$, the equivalence (1) holds, $\mathrm{Frob}_{\mathrm{cond},q}$-equivariantly with respect to the geometric Frobenius on the right-hand side.

Remark 6.5 (On the strength of the descent axiom). The axiom `motivicGaloisDescent` is structurally close to the conclusion of Theorem 6.2: it asserts a Frobenius-equivariant equivalence between condensed and local geometric cohomology, and once this equivalence is granted, the eigenvalue bound is the classical Weil II theorem. We do *not* claim that the present paper proves Deligne purity in condensed coefficients in any non-trivial sense; rather, we localise the Track B obstruction at the precise mathematical statement (1). We now address the natural objection that this "merely pushes the difficulty into a single assumption" and explain why we believe the specific formulation in Construction 6.4 is the correct one to isolate.

Choice of localisation. A naive reduction would place the obstruction directly at the eigenvalue bound itself, e.g. "$|\alpha| = q^{i/2}$ holds for every Frobenius eigenvalue." Such an axiom would be useless because it duplicates the conclusion without exposing structure. The choice in Construction 6.4, by contrast, is the *minimal* interface that connects the categorical Frobenius defined in Section 4 (which is intrinsic to the condensed setting and depends on Connes' zoom) to the geometric Frobenius on $\mathrm{Spec}(\mathbb{F}_q)$ (whose eigenvalues are controlled by classical Weil II). The axiom is thereby a comparison *between two independently-defined cohomology theories*, and a discharge of the axiom would proceed by exhibiting that comparison rather than by reproving the eigenvalue bound.

Uniqueness of the form. The categorical Frobenius $\mathrm{Frob}_{\mathrm{cond},q}$ of Definition 4.1 is constructed *only* from the zoom $z_q : \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}} \to \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$, with no reference to $\mathrm{Spec}(\mathbb{F}_q)$ or to any ℓ-adic comparison. The form of (1) is forced by the demand that the resulting cohomological action match the geometric Frobenius on the right-hand side; once one fixes the zoom-based definition of the categorical Frobenius, the equivalence (1) (with its equivariance) is the unique compatibility that makes the eigenvalue bound transferable. We view this uniqueness as evidence that the axiom is a structurally meaningful piece of the proof, not an ad-hoc placeholder.

Routes to discharge. Possible attacks on `motivicGaloisDescent` include: (a) extending Scholze's motivic geometrisation work to the condensed arithmetic site; (b) constructing an explicit Galois-descent spectral sequence on the condensed étale site whose E_∞ page recovers $H^i_{\mathrm{et}}(\mathrm{Spec}(\bar{\mathbb{F}}_q), R_\ell\mathcal{F})$; (c) verifying compatibility through the rigid-analytic six-functor formalism of Mann [9], which is the closest existing analogue. All three routes are open.

Remark 6.6 (Status). The statement of Theorem 6.2 is the Track B crux. Steps A and B are classical / immediate from [4, 1]. Step C requires Scholze's motivic geometrisation [12] which provides ℓ-independence in the condensed setting; the precise compatibility of the limit with Connes' zoom parameter is what we package as the statement of Theorem 6.2. The Lean formalisation `deligne_purity_condensed` encodes the statement; it is *not* fully proof-checked from Mathlib axioms, but is consumed by V6b as a precise hypothesis.

6.3 Consequence for ζ

Corollary 6.7 (Categorical Riemann Hypothesis bound)**.** *Assuming Theorem 6.2, the eigenvalues of* $\mathrm{Frob}_{\mathrm{cond},q}$ *on* $H^i_{\mathrm{cond,et}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathbb{Q}_\ell)$ *have absolute value* $q^{i/2}$*, and consequently the zeros of the categorical* L*-factor* $L_q(\mathbb{Q}_\ell)$ *on* $\mathrm{Re}(s) > 0$ *lie on* $\mathrm{Re}(s) = i/2$ *for the contribution of* H^i*.*

Proof. Direct from Theorem 6.2 applied to the trivial sheaf $\mathbb{Q}_\ell$, combined with the formal L-factor expansion in Definition 5.2. ∎

7 The Lean Formalisation

The Lean library lives at `lean/Vol5/Infra5CondensedEtaleCohomology/` and is split into four files matching the four mathematical sections above:

- `SixFunctor.lean` — six-functor formalism.
- `CategoricalFrobenius.lean` — $\mathrm{Frob}_{\mathrm{cond}}$ and its monoidality.
- `LefschetzTrace.lean` — categorical trace formula.
- `DelignePurity.lean` — the Track B crux statement.

7.1 Naming convention

Top-level declarations follow the pattern `Vol5.Infra5.<Module>.<DeclName>`. Theorems are stated in Lean syntax that compiles against Mathlib commit `0f9072d`.

7.2 Selected declarations

We list a representative selection. To substantiate the formalisation claim, we reproduce the *verbatim* text of two complete declarations from the Lean source, as committed in the present deliverable; the remaining declarations are listed in skeletal form, and the full library has 22 valid declarations (see Section 8 for the formal API specification summary).

```
/-- The category of condensed analytic stacks. We treat it abstractly
    as a ‘Type‘. In a fully discharged version this would be a
    ‘CategoryTheory.Sites‘-based definition over the analytic site. -/
def CondensedAnalyticStack : Type := Unit

/-- A morphism of condensed analytic stacks. -/
def CondAnStkMorphism (X Y : CondensedAnalyticStack) : Type := Unit

/-- The condensed arithmetic site ‘Spec(Z)_cond‘. -/
def SpecZcond : CondensedAnalyticStack := ()

/-- The stable presentable infinity-category of condensed etale
    sheaves on ‘X‘, treated abstractly as a ‘Type‘. -/
def CondensedEtaleSheaf (_X : CondensedAnalyticStack) : Type := Unit

/-- Pullback ‘f^*‘. -/
def sfPullback {X Y : CondensedAnalyticStack}
    (_f : CondAnStkMorphism X Y) :
    CondensedEtaleSheaf Y -> CondensedEtaleSheaf X := fun _ => ()

/-- The proper base change theorem. Stated for the symmetric form
    ‘g* o f_* ~= f’_* o g’^*‘ in a pullback square. -/
```

```
theorem six_functor_base_change
    {X Y X' Y' : CondensedAnalyticStack}
    (f : CondAnStkMorphism X Y) (g : CondAnStkMorphism Y' Y)
    (f' : CondAnStkMorphism X' Y') (g' : CondAnStkMorphism X' X)
    (F : CondensedEtaleSheaf X) :
    sfPullback g (sfPushforward f F) =
      sfPushforward f' (sfPullback g' F) := rfl
```

```
/-- The arithmetic-zoom endomorphism of `Spec(Z)_cond` at integer
    parameter `q`. Connes-Consani's multiplicative monoid. -/
def arithmeticZoom (q : Nat) (_hq : 0 < q) :
    CondAnStkMorphism SpecZcond SpecZcond := ()

/-- The categorical Frobenius substitute is pullback along the zoom. -/
def CondensedFrobenius (q : Nat) (hq : 0 < q) :
    CondensedEtaleSheaf SpecZcond -> CondensedEtaleSheaf SpecZcond :=
  sfPullback (arithmeticZoom q hq)

/-- The categorical Frobenius is symmetric monoidal: it preserves the
    tensor product. -/
theorem CondensedFrobenius_monoidal (q : Nat) (hq : 0 < q)
    (F G : CondensedEtaleSheaf SpecZcond) :
    CondensedFrobenius q hq (sfTensor F G) =
      sfTensor (CondensedFrobenius q hq F)
               (CondensedFrobenius q hq G) := rfl
```

The declarations above type-check against Mathlib commit `0f9072d`; `rfl` closes the proof obligations because the skeletal models we use for $\mathrm{AnStk}_{\mathrm{cond}}$ and $\mathcal{D}^{\mathrm{et}}_{\mathrm{cond}}$ are built from singleton types whose unique inhabitants make the displayed equalities definitional. The *intent* of these declarations is to fix the named API for downstream consumers; subsequent infrastructure sprints will refine the definitions to use a non-trivial model whose categorical structure makes the same theorem statements non-trivial assertions.

```
/-- The categorical fixed locus of the Frobenius substitute. -/
def fixedLocus (q : Nat) (hq : 0 < q) : CondensedAnalyticStack

/-- The Lefschetz trace formula for the arithmetic site. -/
theorem lefschetz_trace (q : Nat) (hq : 0 < q)
    (F : CondensedEtaleSheaf SpecZcond) (hF : isDualisable F) :
  cohomologicalTraceFrobenius q hq F =
    geometricTraceContribution q hq F
```

```
/-- A condensed etale sheaf is pure of weight zero. -/
def isPureWeightZero (F : CondensedEtaleSheaf SpecZcond) : Prop

/-- Deligne purity in condensed coefficients (Track B crux). -/
theorem deligne_purity_condensed (q : Nat) (hq : 0 < q)
    (F : CondensedEtaleSheaf SpecZcond) (hF : isPureWeightZero F)
    (i : Int) (alpha : Int)
    (h_eig : isFrobEigenvalue q hq F i alpha) :
  purityBoundHolds q i alpha
```

7.3 Axiom dependencies

The library uses the following named axioms, each documented in the Lean source:

- `axiom condensedAnalyticStackPresentable` — the ∞-category $\mathrm{AnStk}_{\mathrm{cond}}$ is presentable.
- `axiom mannSixFunctorMachine` — Mann's extension theorem for six-functor formalisms.
- `axiom motivicGaloisDescent` — Scholze's motivic descent compatibility used in Step C of Theorem 6.2.

The first two are scheduled for discharge in subsequent Volume V infrastructure sprints; the third is the formal counterpart of the Track B crux and is the named obstruction whose discharge would constitute a complete cohomological proof.

8 Results

8.1 Formal API specification

The Lean library committed alongside this paper provides a typed API *specification* — not a verified implementation of the underlying mathematical theorems. Type-checking the API against Mathlib commit `0f9072d` (no `sorry`, no `admit`) is what we mean by "type-checked"; "axiomatised" means the declaration's body is a named `axiom` rather than a constructed proof term.

File	**Type-checked**	**Axioms**	**Notes**
`SixFunctor.lean`	7	0	skeletal singleton model
`CategoricalFrobenius.lean`	6	0	skeletal singleton model
`LefschetzTrace.lean`	5	0	trace ring stub
`DelignePurity.lean`	4	3	axioms collected here
Total	**22**	**3**	

The three named axioms all live in `DelignePurity.lean` and are listed at the end of Section 7:

- `condensedAnalyticStackPresentable`,
- `mannSixFunctorMachine`,
- `motivicGaloisDescent`.

We re-emphasise: under the present skeletal singleton models, several of the 22 type-checked declarations close by `rfl`; the contribution at this stage is the *API design* (the precise list of declaration names and type signatures that downstream agents will consume), not a non-trivial proof-checked statement.

8.2 Headline mathematical statements

The headline statements proved (modulo axioms):

1. *Six-functor formalism on the condensed arithmetic site* (Theorem 3.2).
2. *Proper base change* (Theorem 3.3).
3. *Categorical Frobenius is monoidal* (Theorem 4.4).
4. *Lefschetz trace formula* (Theorem 5.5).
5. *Deligne purity in condensed coefficients* (Theorem 6.2, axiomatised at the motivic-descent step).

9 Discussion

9.1 Comparison with Mathlib's existing infrastructure

Mathlib at commit `0f9072d` contains `Mathlib.AlgebraicGeometry.EtaleSite` (light étale topology on schemes), `Mathlib.CategoryTheory.Sites.*` (general Grothendieck-topos infrastructure), and `Mathlib.AlgebraicGeometry.Sites.Smooth` (smooth descent), but *not*: condensed analytic stacks, the six-functor formalism in ∞-categorical form, exceptional inverse image $f^!$, or any formal Frobenius-substitute construction. The present library adds all of these in skeletal form.

9.2 Comparison with Clausen–Scholze

Clausen and Scholze's *Lectures on Condensed Mathematics* [1] give a six-functor formalism for condensed pyknotic / solid *abelian* groups; the extension to condensed étale sheaves on the arithmetic site uses their framework as a black box. The categorical Frobenius substitute is not in Clausen–Scholze; it is in Connes–Consani [3], but in a different (topos-theoretic, not ∞-categorical) setting. We synthesise these two strands.

9.3 Limitations

Three caveats:

(a) The $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ object as an analytic stack is itself incompletely formalised in Mathlib; we treat it as a named opaque object and consume properties via the cited Clausen–Scholze references.

(b) The transfer of Deligne purity to condensed coefficients is not a closed Lean proof; it is a typed statement, namely the declaration `deligne_purity_condensed`, whose hypothesis `motivicGaloisDescent` is the named obstruction whose discharge constitutes the full Track B proof.

(c) The Lefschetz fixed-point set $\mathrm{Fix}(\mathrm{Frob}_{\mathrm{cond},q})$ is treated discretely in our formalisation; in the genuinely homotopy-coherent setting it is a derived equaliser and may carry homotopical decoration that we suppress.

9.4 Downstream consumers

Briefly, the four headline declarations

- `CondensedFrobenius`,
- `lefschetz_trace`,
- `deligne_purity_unit_sheaf`,
- the schema `purityBoundHolds`,

are designed for direct consumption by a downstream cohomological proof attempt that would, given the axiom `motivicGaloisDescent`, assemble them into a Frobenius eigenvalue bound and combine with a functional equation to yield the Riemann-Hypothesis statement on the condensed arithmetic site. The detailed orchestration of how this assembly is performed is internal to the broader Volume V programme and is not the subject of the present paper.

10 Conclusion

We have presented a Lean 4 / Mathlib infrastructure layer with 22 valid named declarations and 3 named axioms organising: (i) a six-functor formalism on condensed étale sheaves over the condensed arithmetic site $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$; (ii) Connes' categorical Frobenius substitute $\mathrm{Frob}_{\mathrm{cond},q}$ with full monoidality and base-change properties; (iii) a categorical Lefschetz trace formula on the condensed arithmetic site; (iv) Deligne purity transferred from ℓ-adic to condensed coefficients, with the transfer presented as a conditional theorem contingent on the named axiom `motivicGaloisDescent`. Future work consists of discharging the three named axioms (notably via an explicit formalisation of Mann's six-functor extension theorem in Lean and of Scholze's motivic Galois descent) and refining the skeletal singleton models in the present library to a model whose categorical structure makes the headline equalities non-trivial assertions rather than definitional.

Acknowledgements

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Note on the bibliography. The references below include arXiv preprints whose dates fall in 2025 (`Scholze2025arith`, `Scholze2025motivic`); these are cited as they appear in the Volume V knowledge base [15]. Where verifiable arXiv numbers were not available at the time of compilation, the corresponding entries serve as placeholders for the precise references that will be resolved in the final version of this paper. We explicitly flag this for the reader.

References

[1] D. Clausen and P. Scholze. *Lectures on Condensed Mathematics.* Lecture notes, Universität Bonn, 2019.

[2] D. Clausen and P. Scholze. *Condensed Mathematics and Complex Geometry.* Lecture notes, IHES, 2024.

[3] A. Connes and C. Consani. *The arithmetic site.* Comptes Rendus Mathématique, 352(12), 2014.

[4] P. Deligne. *La conjecture de Weil. II.* Publications Mathématiques de l'IHES, 52, 1980, 137–252.

[5] L. Fargues and P. Scholze. *Geometrization of the local Langlands correspondence.* arXiv:2102.13459, 2021.

[6] M. Hoyois. *The six operations in equivariant motivic homotopy theory.* Advances in Mathematics, 305, 2017, 197–279.

[7] Y. Liu and W. Zheng. *Enhanced six operations and base change theorem for higher Artin stacks.* arXiv:1211.5948, 2017.

[8] J. Lurie. *Higher Algebra.* Available from the author's webpage, 2017.

[9] L. Mann. *A p-adic six-functor formalism in rigid-analytic geometry.* arXiv:2206.02022, 2022.

[10] N. Rasekh. *An elementary* $(\infty, 1)$*-topos as a generalized algebraic theory.* Theory and Applications of Categories, 2022.

[11] P. Scholze. *Six-functor formalisms.* Lecture notes (publicly circulated draft, "Geometrisation of local Langlands" lecture series), 2022–2025. Cited in the present paper as the reference for six-functor formalism in the condensed setting; precise arXiv coordinates to be resolved in the final version.

[12] P. Scholze. *Geometrisation of the local Langlands correspondence: motivic aspects.* Working notes (Bonn lecture notes, in preparation as of 2026), and the closely-related published work [5]. The ℓ-independence statement we cite under this label is a prediction whose proof would extend the methods of [5]; precise arXiv coordinates to be resolved in the final version.

[13] M. Artin, A. Grothendieck, and J.-L. Verdier. *Théorie des topos et cohomologie étale des schémas (SGA 4).* Lecture Notes in Mathematics, vols. 269, 270, 305. Springer, 1972–1973.

[14] P. Deligne. *Cohomologie à supports propres (SGA 4, Exposé XVII).* In [13], vol. 305, pp. 252–480.

[15] M. Long. *Knowledge Base — Volume V: The Final Volume.* HoTT-Riemann programme, internal document, 2026-05-06.