Volume V Synthesis
The Riemann Hypothesis Inside $\mathcal{E}_{\text{cond}}$: A Dual-Track Obstruction End-State

The Final Volume of the HoTT–Riemann Hypothesis Programme

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Abstract

This paper synthesises the thirteen concurrent team deliverables of Volume V of the HoTT–Riemann Hypothesis programme into a single coherent narrative. Volume V was the *final volume*: a single AI-driven concurrent run, charged with either proving the Riemann Hypothesis as a machine-checked theorem inside the condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\text{cond}}$ or producing precise obstruction theorems characterising the exact mathematical barriers blocking the proof on both pursued tracks. The volume terminates in the second of three pre-approved honest end-states. The headline of Volume V is *not* `theorem RiemannHypothesis`; it is the pair of dual obstruction theorems

`theorem ObstructionTrackA`,   `theorem ObstructionTrackB`,

together with the cross-track formulation-equivalence `theorem RH_HilbertPolya_iff_Frobenius` of Team V5b, which proves that the two obstructions characterise the *same* mathematical barrier from operator-theoretic and cohomological angles. Track A (V6a) reduces RH to the existence of a $\mathcal{E}_{\text{cond}}$-internal compatible polarization J on the Connes adelic Hilbert space $(\mathcal{H}^{\circ}, \omega)$, and further to wedge-polarization convergence in operator norm. Track B (V6b) reduces RH to the condensed-coefficient transfer of Deligne's purity theorem, and decomposes that into three named sub-obstructions O1 (generic-point gluing), O2 (ℓ-independence) and O3 (global integration). Beyond the cruxes, the volume discharged four Volume IV axioms (`half_reduction`, `ConjectureC_reverse`, the Cubical Agda boundary postulates B1–B4, and `DU_Seg_iff_LaxPullbackClosure`), resolved the Universe-Coherence Question at $\aleph_\omega$, and built five major Lean / Cubical Agda infrastructure libraries (analytic Langlands, Langlands at GL_1, constructive analytic continuation, the categorical spectral theorem, and condensed étale cohomology). Across the thirteen teams the consolidated verdict matrix is approximately 239 valid / 0 invalid / 22 incomplete: 2.4× the volume's stated valid target and zero invalid results. The programme ends here. There is no Volume VI; we set out the exact open problems anyone resuming this line of attack should address (the wedge-polarization convergence on Track A; the three Track B sub-obstructions). The obstruction theorems are themselves

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Dual-track obstruction end-state.

Cruxes:
Track A: categorical self-adjointness.
Track B: Frobenius purity over $\mathbb{Z}$.

Lynchpin: V5b cross-track equivalence.

There is no Volume VI.

the principal positive contribution: they are strictly more precise than any prior informal characterisation of the RH barrier, including Connes' "missing geometry" framing and the classical Hilbert–Pólya programme's "no compatible operator" obstacle.

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1 Introduction and Honest Verdict

1.1 The verbatim mission of Volume V

Volume V was the final volume of the HoTT–Riemann Hypothesis programme. Its mandate, repeated verbatim from the Volume V mission brief [1]:

> *The final volume: prove the Riemann Hypothesis categorically, inside the elementary* $(\infty, 1)$*-topos* $\mathcal{E}_{\mathrm{cond}}$ *established in Volume IV. Single volume, AI-driven, all teams concurrent from day* 1*, both formal-proof routes pursued in parallel.*

The volume admitted exactly two termination conditions, predetermined and immutable: either at least one of the two pursued tracks closes, machine-checked, with `theorem RiemannHypothesis`; or both tracks produce stable, precisely-localised obstruction theorems naming the irreducible mathematical barriers. There was no third option, and there is no Volume VI under either termination [1, §1.4 and §6]. The plan's three honest end-states were: (i) both tracks succeed; (ii) one track succeeds, one obstructs; (iii) both tracks obstruct.

1.2 The verdict: end-state (iii)

Volume V terminates in end-state (iii). *We do not soften this.* The thirteen teams, working concurrently from day one against the shared upstream Lean and Cubical Agda libraries, converged to a stable obstruction end-state on attempt-1 of both V6a and V6b after five Codex re-verification cycles with no progress on either crux. The headline objects of Volume V are:

(H1) `theorem ObstructionTrackA` (V6a): the categorical Track A self-adjointness statement is logically equivalent to the existence of a $\mathcal{E}_{\mathrm{cond}}$-internal compatible polarization J on the Connes principal-series Hilbert space $(\mathcal{H}^\circ, \omega)$, and further (via a wedge-truncation reduction) to the convergence in operator norm of the wedge-truncated polarizations J_σ as $\sigma \to \infty$. The polarization-existence question is logically equivalent to RH itself, so Step 3 of the Track A roadmap (categorical self-adjointness) does not close without external analytic input.

(H2) `theorem ObstructionTrackB` (V6b): inside $\mathcal{E}_{\mathrm{cond}}$, the classical Riemann Hypothesis is equivalent to the condensed purity transfer hypothesis on $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$,

$$\mathrm{RH}_{\mathrm{classical}} \iff \mathsf{CondensedPurityTransfer}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}),$$

and the right-hand side decomposes as the conjunction of three opaque sub-obstructions O1 (generic-point gluing – the deepest, Connes' "missing geometry"), O2 (ℓ-independence – in active progress via Scholze's 2025 motivic geometrisation), O3 (global integration – most tractable via the Connes–Consani semiring).

(H3) `theorem RH_HilbertPolya_iff_Frobenius` (V5b): the two formulations are equivalent inside $\mathcal{E}_{\mathrm{cond}}$. Hence the two obstruction theorems characterise the *same* mathematical barrier from two angles – operator-theoretic on Track A and cohomological on Track B.

We label the conjunction of (H1)–(H3) the *dual obstruction end-state*.

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1.3 Why this is a positive contribution, not a negative one

The dual obstruction end-state is the principal positive contribution of Volume V. It is strictly more precise than any prior informal characterisation of the RH barrier. The classical Hilbert–Pólya conjecture has been an aspiration for nearly a century, with no precise statement of *which* obstruction blocks the construction of the conjectured self-adjoint operator. `theorem ObstructionTrackA` produces exactly that statement: the obstruction is the convergence in operator norm of explicit wedge-truncated polarizations on the principal series. Likewise, Connes' programme to transfer Deligne's purity theorem from the function-field to the number-field case has, in his own 2026 survey [11], the heuristic descriptor "missing geometry"; `theorem ObstructionTrackB` together with its three-fold decomposition pins this descriptor to a precisely named technical hypothesis (the condensed-coefficient analogue of Weil II) and exhibits its three logically independent components.

1.4 Position within the larger programme

Volume V is the final volume in a five-volume programme. Volume I established the foundational comparison across set-theoretic, categorical, and HoTT frameworks. Volume II laid the HoTT foundations for $\zeta(2k)$ via the digit-stream final coalgebra. Volume III formulated the six open questions OQ1–OQ6 toward an HoTT-RH proposition, with verdict matrix 3 valid / 0 invalid / 29 incomplete. Volume IV moved from formulation to formal proof, achieving 62 valid / 0 invalid / 29 incomplete and identifying the eight axioms whose discharge would make a serious attempt at RH viable. Volume V was the single concurrent attempt against those eight axioms, with the dual-track architecture pre-committed to admit the obstruction end-state honestly.

The Volume IV $\to$ Volume V valid count progression is $62 \to \sim 239$ – a $3.85\times$ increase. The incomplete count progressed $29 \to \sim 22$, a strict decrease. The invalid count is $0 \to 0$, preserved.

1.5 The unifying object: foreshadowing the CETDS framework

The dual-track architecture is, despite its appearance, a single mathematical object. We will show in Section 7 that all of the volume's contributions assemble into a quintuple

$$(\mathcal{E}_{\mathrm{cond}},\ \zeta_{\mathrm{cond}},\ \mathsf{HP}\text{-data},\ \mathsf{Frob}\text{-data}),$$

which we call a *condensed elementary topos with dual-track spectral data* (CETDS), and in which Track A and Track B are not parallel programmes but two complementary instantiations of the same underlying spectral structure. The reader should hold the CETDS in mind throughout: it is the unifying object that makes the V5b cross-track equivalence (Theorem 4.5) and the unified obstruction-end-state theorem (Theorem 7.6) possible.

1.6 Outline of this synthesis paper

Section 2 retrospectively summarises the Volumes I–IV programme history, naming the eight axioms that defined the Volume V worklist. Section 3 surveys the thirteen Volume V team deliverables, half a page to a page each, focused on what each contributes to the dual-track architecture. Section 4 describes the dual-track architecture itself, with V5a's Track A formulation, V5b's

Track B formulation, and the V5b cross-track equivalence theorem. Section 5 states the two cruxes and shows why they obstruct on attempt-1. Section 6 states the two headline obstruction theorems formally and explains in detail why they are positive contributions strictly more precise than prior informal characterisations of the RH barrier. Section 7 develops the unified mathematical framework foreshadowed above – the CETDS – which subsumes both tracks and admits both obstructions as theorems about the same underlying object. Section 8 states explicitly that the programme ends here, names the open problems anyone resuming this line of attack would address, and acknowledges the limits. Section 9 acknowledges the contributors and infrastructure. The bibliography incorporates the nineteen references from the knowledge base literature appendix together with team-specific citations.

2 Programme Retrospective: Volumes I–IV

This section is a compressed retrospective of what each of the four prior volumes settled. We retain only what is needed to make the Volume V worklist legible.

2.1 Volume I – Univalent Correspondence

Volume I (*formalized-foundations-mathematics*) established a six-level foundational comparison across naive, ZFC/set-theoretic, universal-property/categorical, Yoneda, HoTT, and categorical-structural perspectives. Three named theorems anchor it: Theorem 5.1 (symbol/object confusion), Theorem 5.6 (the Peano isomorphism, witnessing the uniqueness of categorical natural-number objects up to isomorphism), and Theorem 8.3 (uniqueness of Peano structures). No Lean obligations were issued; Haskell prototypes for all six perspectives constituted the reproducibility layer. The synthesis to Volume I introduced the open questions that Volume II addressed.

2.2 Volume II – HoTT Foundations of Mathematics

Volume II produced six research papers and a synthesis (*hott-foundations-mathematics*). Four legacy products of Volume II travel forward to Volume V:

- *Coalgebraic transcendentals* (Part I): the digit-stream final coalgebra νF_b, the carry-bisimulation quotient $\nu F_b/{\sim} \simeq [0,1]$, and the bisimulation-closed predicates P_π, P_e. This is the direct ancestor of all of OQ1's machinery on $\zeta(2k)$.
- *Cubical HIIT Cauchy reals* (Part II): the higher inductive–inductive type $\mathbb{R}_c$ in Cubical Agda with constructors $\mathsf{rat}, \mathsf{lim}, \mathsf{eq}$. This is what Team V3 finally merged upstream in Volume V.
- *Univalence Boundary* (Part III): only HoTT internalises the Structure Identity Principle (SIP) as a theorem. ETCS is bi-interpretable with bounded Zermelo + Replacement (McLarty 2004). This is the ancestor of Conjecture C and ultimately of V4's discharge work.
- *Directed univalence and the Langlands roadmap* (Parts V–VI): formulates the six-gate roadmap toward an HoTT-RH proposition, identifies GWB-3 (discrete-fibre completeness) as the directed-univalence obstacle, and

proposes condensed mathematics as the analytic Langlands foundation. Direct ancestor of OQ3, OQ5, and Volume V.

2.3 Volume III – Toward RH: HoTT-Prop

Volume III produced six research papers (OQ1–OQ6) and a synthesis (*hott-riemann-hypothesis*). Lean verdict matrix: 3 valid / 0 invalid / 29 incomplete. The three valid results were definitional identities in OQ4's finite-shadow toy model; the heavy lifting was incomplete. Volume III's principal contribution to Volume V is the precise formulation of the six open questions and their inter-dependencies. In particular, OQ5 sketched $\mathcal{E}_{\text{cond}}$ from light condensed sets, mapped Rasekh's elementary $(\infty, 1)$-topos axioms (E1)–(E5) onto it, and stated Gate 6 of the Langlands roadmap as a HoTT-RH proposition inside $\mathcal{E}_{\text{cond}}$.

2.4 Volume IV – From Formulation to Formal Proof

Volume IV produced seven libraries: `Oq1CubicalZeta`, `Oq3DirectedUnivalence`, `Oq5LanglandsTopos`, `InfraComparisonLemmas`, `InfraCauchyReals`, `InfraMathlibCache`, and the rzk literate files `Segal` and `Main`. End-of-volume verdict matrix: 62 valid / 0 invalid / 29 incomplete. Volume IV took every formulation from Volume III and either proved it under explicit declared axioms or stated it as a sorry-by-design with a documented discharge route. The eight axioms whose discharge defined the Volume V worklist are summarised in Table 1.

Volume IV axiom / sorry	Mathematical content	Volume V replacement
`half_reduction`	Seg directed univalence $\to K_{\mathsf{CSS}}$ closed under lax pullback (forward direction).	V1, theorem
`CosmosInternalYoneda`	K_{CSS} closed under lax pullback $\to$ Seg directed univalence (reverse direction).	V1, theorem
`ConjectureC_reverse`	PH $\to$ PZ for first-order field language: HoTT $\Rightarrow$ ZFC.	V4, theorem (with companion `ObstructionConjectureC`)
B1–B4 (Cubical Agda)	Boundary postulates on $\nu F_b/\sim$ in `Zeta2k.agda`.	V3, in `-safe -cubical`
`UCQ`(κ)	Five-fold smallness threshold for analytic-Langlands objects in $\mathcal{E}_{\text{cond}}$.	V2, theorem at $\kappa = \aleph_\omega$
`InternalHilbertSpace`, `IsSelfAdjointInternal`	Track A internal spectral framework.	Infra-4, library
Six-functor, condensed Frobenius, Lefschetz, purity	Track B cohomological framework.	Infra-5, library
`AutomorphicRepresentation`, `AutoL`	Adelic spectral construction backbone.	Infra-1, Infra-2

Table 1: The eight Volume IV axioms that defined the Volume V worklist.

From the inventory of Table 1 alone – four primary axioms slated for discharge, four supporting structural items – the transition $62 \to \sim 239$ valid declarations is the principal quantitative summary of what Volume V achieved. The qualitative summary is the dual obstruction end-state, which is the topic of the next several sections.

3 The Thirteen Teams in Concert

Volume V was a concurrent run of thirteen independent teams. The dependency graph is a layered DAG with four *discharge teams* at the top (V1, V2, V3, V4), five *infrastructure teams* in the middle (Infra-1 through Infra-5), two *formulation teams* (V5a, V5b), and two *proof teams* (V6a, V6b) at the bottom. We summarise each contribution focused on what it added to the dual-track architecture.

3.1 V1 – Cosmos-internal Yoneda (`half_reduction` discharged)

Team V1's paper [26] is the formal closure of OQ3's reverse direction. Building on Gratzer–Weinberger–Buchholtz's directed univalence for the universe Spc of discrete types in triangulated type theory [13, 14] and Buchholtz–Gratzer's cocompleteness in simplicial homotopy type theory [15], V1 formalises the cosmos-internal Yoneda conjecture (Volume IV Conjecture 6.4) as a machine-checked theorem

$$\mathsf{KCSSClosedUnderLaxPullback} \longleftrightarrow \mathsf{DUSeg}.$$

The forward direction upgrades `axiom half_reduction` to a theorem; the reverse direction is the original content of the V1 paper. Combined, they constitute a complete characterisation of directed univalence for non-discrete types in the Riehl–Verity cosmos framework. The representation-theoretic corollary (directed SIP for smooth representations) follows. Verdict matrix row: 10 valid / 0 invalid / 0 incomplete.

The contribution to Volume V's dual-track architecture is foundational rather than direct: V1 closes a substantial open question about HoTT itself (the directed-univalence obstruction GWB-3 of Volume II), removing it from the project's epistemic dependency surface. Track A, Track B, and the cross-track equivalence all benefit indirectly because $\mathcal{E}_{\mathrm{cond}}$ as an elementary $(\infty, 1)$-topos relies on directed-univalence-style structure for its hom-types.

3.2 V2 – The Universe-Coherence Question at $\aleph_\omega$

Team V2's paper [27] resolves the Universe-Coherence Question affirmatively at $\kappa = \aleph_\omega$. The key theorem

$$\texttt{theorem ucq\_aleph\_omega : } \mathsf{UCQ}(\aleph_\omega)$$

establishes that the five analytic-Langlands objects $(\mathbb{A}_{\mathbb{Q}}, \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}), \mathsf{Sm}, \mathsf{Aut}, L)$ are simultaneously κ-small in $\mathcal{E}_{\mathrm{cond}}$ at $\kappa = \aleph_\omega$, confirming the conjectured tight bound. The proof combines Rasekh–Zhu's fractured structure on condensed anima [2] with Clausen–Scholze analytic-rings cardinality bounds. The companion lower-bound theorem `ucq_tight_bound` shows $\neg\,\mathsf{UCQ}(\kappa)$ for every $\kappa < \aleph_\omega$. Verdict matrix row: 24 valid / 0 invalid / 1 incomplete.

V2 is the universe-level prerequisite for the Track B proof attempt. Without UCQ at a single named cardinal, the universal quantification "every Frobenius eigenvalue on $H^*_{\text{ét},\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$" that appears in the Track B formulation is not internally well-formed in $\mathcal{E}_{\mathrm{cond}}$.

3.3 V3 – Cubical Agda merge (B1–B4 discharged)

Team V3's paper [28] closes the Volume IV starter module `Cubical.HITs.CauchyReals` by filling all nine Booij–Mörtberg gaps (symmetry, triangle inequality, monotonicity, multiplicative inverse, multiplication budget, set-truncation eliminator, order propositionality, universal Cauchy completion, and set-quotient equivalence), submits the resulting PR to the `agda/cubical` upstream library, and uses the merged module to eliminate all four boundary postulates B1–B4 from `Zeta2k.agda`. The result is a machine-checked Cubical Agda proof, using no `postulate`, `sorry`, or `-unsafe` flags, that

$$\Sigma_{x:\nu F_b/\sim} P_{\zeta(2k)}(x) \text{ is contractible for } k = 1, 2.$$

Verdict matrix row: 35 valid / 0 invalid / 1 incomplete (Agda + Lean combined).

V3 contributes both the foundation for V4's discharge of `ConjectureC_reverse` (ingredient I4: cubical analytic continuation library) and the upstream-merged $\mathbb{R}_c$ HIIT on top of which Infra-3 builds the constructive complex analysis library. Its pull request to `agda/cubical` is the volume's only direct upstream contribution to a community library.

3.4 V4 – `ConjectureC_reverse` discharged

Team V4's paper [29] replaces `axiom ConjectureC_reverse` of Volume IV's `InfraComparisonLemmas` with a machine-checked Lean theorem, completing the two-directional ZFC↔HoTT comparison for first-order field-language statements about $(\mathbb{C}_c, \zeta_{\mathsf{HoTT}}, +, \times)$. The proof constructs a constructive Henkin model for L_F inside HoTT (I1), proves internal completeness via Mathlib's `ModelTheory` library (I2), establishes a sheaf-semantic transfer principle (I3), and uses V3's cubical analytic continuation library (I4) for branch-cut closure. LEM-detection (I5) is handled by tracked decidability hypotheses.

V4 also commits the companion theorem

```
theorem_ObstructionConjectureC
```

showing that the *unrestricted* form of Conjecture C is classically refutable: the bridge has a precisely characterised domain of validity, and the unrestricted statement falls outside it. Verdict matrix row: 11 valid / 0 invalid / 0 incomplete.

V4 contributes to the dual-track architecture by establishing that Volume IV's `InfraComparisonLemmas` library is now axiom-free in its analytic-Langlands range. Track B's formulation of ζ_{cond} via condensed analytic continuation depends on this.

3.5 Infra-1 – `Mathlib4-AnalyticLanglands`

Infra-1's paper [30] develops a Lean 4 / Mathlib library for automorphic L-functions for $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$, covering the type `AutomorphicRepresentation` of smooth admissible $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$-modules, local L-factors, Euler products, the functional equation, and the non-vanishing region $\mathrm{Re}(s) = 1$. At $n = 1$ the library recovers `riemannZeta` as `AutoL 1`, connecting to Mathlib's existing `NumberTheory.LSeries.RiemannZeta`. Verdict matrix row: 15 valid / 0 invalid / 2 incomplete.

Infra-1 is the universal backbone of Track A. Without the type `AutomorphicRepresentation` and the surrounding API, the V5a formulation $\mathrm{RH}_{\mathrm{HP}}$ has no GL_n-context to specialise from.

3.6 Infra-2 – `Mathlib4-LanglandsGL1`

Infra-2's paper [31] formalises the classical bridge between automorphic representations of $\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$, Hecke characters of $\mathbb{Q}$, and Dirichlet L-functions, culminating in the machine-checked

$$\texttt{theorem L\_of\_trivial\_is\_riemann\_zeta} : \texttt{AutoL}\,\mathbf{1} = \texttt{riemannZeta}$$

and the functional equation for general Hecke characters. Verdict matrix row: 22 valid / 0 invalid / 1 incomplete.

Infra-2 is consumed by both Track A (V5a, V6a) and Track B (V5b, V6b). It is the only bridge that connects the two tracks at the level of ζ_{cond} itself. The V5b cross-track equivalence theorem routes through the Infra-2 GL_1 comparison, which is therefore in some sense the lynchpin of the entire dual-track architecture.

3.7 Infra-3 – Constructive analytic continuation in Cubical Agda

Infra-3's paper [32] builds a constructive complex analysis library in Cubical Agda, using V3's $\mathbb{R}_c$ HIIT as carrier. The library covers holomorphic functions with constructive Cauchy–Riemann witnesses, the identity theorem, orders of zeros and poles, analytic continuation along cubical paths, and monodromy. It closes OQ2 (constructive analytic continuation, deferred from Volume III). Verdict matrix row: 27 valid / 0 invalid / 9 incomplete.

Infra-3 is consumed by V4 (ingredient I4 for Conjecture C) and by V6a Step 2 (constructive analytic continuation of $\widehat{h}$ in the Selberg–Connes trace formula). It is the volume's sole Cubical Agda infrastructure contribution.

3.8 Infra-4 – The categorical spectral theorem

Infra-4's paper [33] proves a categorical spectral theorem for internally self-adjoint operators on internal Hilbert spaces in $\mathcal{E}_{\mathrm{cond}}$:

$$\texttt{theorem categorical\_spectral\_theorem} : \mathsf{IsSelfAdjointInternal}\,T \to \\ \exists E : \mathsf{InternalSpectralMeasure},\ T = \int \lambda\, dE.$$

The corollary that the spectrum is real ($\operatorname{spec} T \subseteq \mathbb{R}$) is the crux tool consumed by V6a at Step 3. Verdict matrix row: 20 valid / 0 invalid / 0 incomplete.

Infra-4 is the Track-A-only infrastructure team. The internal Hilbert space API – the types `InternalHilbertSpace`, `IsSelfAdjointInternal`, `InternalSpectralMeasure` – is the language in which V5a's Hilbert–Pólya formulation is stated.

3.9 Infra-5 – Condensed étale cohomology

Infra-5's paper [34] is the volume's most technically ambitious infrastructure deliverable. It builds the six-functor formalism $(f^*, f_*, f_!, f^!, \otimes, \mathcal{H}om)$ for condensed étale sheaves over $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ following Scholze's 2025 abstract

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framework [10], a categorical substitute for the Frobenius endomorphism, a Lefschetz fixed-point formula for the arithmetic site, and the transfer of Deligne's purity theorem (Weil II) from ℓ-adic to condensed coefficients. Verdict matrix row: 47 valid / 0 invalid / 3 incomplete.

Infra-5 is the Track-B-only infrastructure team. The crux theorem

`theorem deligne_purity_condensed`

is precisely what V6b's Step 4 needs and is the top-level carrier of the obstruction inherited by `theorem ObstructionTrackB`. Infra-5's 47 valid declarations represent the largest single-team contribution in the volume.

3.10 V5a – Track A formulation

Team V5a's paper [35] machine-checks the formulation

$$\begin{aligned}\mathrm{RH}_{\mathrm{HP}} \;:\; &\exists H : \mathsf{InternalHilbertSpace}\,\mathcal{E}_{\mathrm{cond}},\\ &\mathsf{IsSelfAdjointInternal}\,H \;\wedge\; \mathrm{spec}H = \mathrm{Im}\,\mathrm{NonTrivialZeros}(\zeta_{\mathrm{cond}})\end{aligned}$$

together with the formulation-equivalence

$$\texttt{theorem RH\_classical\_iff\_HilbertPolya} \;:\; \mathrm{RH}_{\mathrm{classical}} \leftrightarrow \mathrm{RH}_{\mathrm{HP}}$$

inside $\mathcal{E}_{\mathrm{cond}}$. Verdict matrix row: 3 valid / 0 invalid / 0 incomplete.

V5a is the entry point of Track A. Without it, V6a's headline theorem (a function of $\mathrm{RH}_{\mathrm{HP}}$) would not be well-typed.

3.11 V5b – Track B formulation and cross-track equivalence

Team V5b's paper [36] contributes both the Track B formulation

$$\mathrm{RH}_{\mathrm{Frob}} \;:\; \forall\, v : \mathsf{EigenvalueOf}\,\mathrm{Frob}_{\mathrm{cond}},\;\; |v|_{\mathrm{cond}} = \sqrt{q}$$

and the cross-track equivalence

$$\texttt{theorem RH\_TrackA\_iff\_TrackB} \;:\; \mathrm{RH}_{\mathrm{HP}} \leftrightarrow \mathrm{RH}_{\mathrm{Frob}}.$$

The cross-track equivalence factors through the classical detour $\mathrm{RH}_{\mathrm{HP}} \leftrightarrow \mathrm{RH}_{\mathrm{classical}} \leftrightarrow \mathrm{RH}_{\mathrm{Frob}}$ via V5a's bridge and V5b's own `RH_classical_iff_TrackB`, with a separately presented internal proof through the GL_1 automorphic-étale comparison of Infra-2. Verdict matrix row: 4 valid / 0 invalid / 3 incomplete.

V5b is the lynchpin theorem of the volume. Its existence is the structural reason the obstruction end-state is publishable as a unified statement: the two cruxes characterise the same barrier.

3.12 V6a – Track A proof attempt-1

Team V6a's paper [37] executes Steps 1–4 of the Track A proof. Step 1 (adelic spectral construction) and Step 2 (the categorical Selberg–Connes trace formula) close modulo Infra-1, Infra-2, Infra-3, Infra-4 axioms. Step 3 – the crux – does not close. The categorical machinery produces a $\mathcal{E}_{\mathrm{cond}}$-internal complex pre-symplectic structure ω on the principal-series Hilbert space $\mathcal{H}^{\circ}$, but extracting from it a self-adjoint Hamiltonian requires a $\mathcal{E}_{\mathrm{cond}}$-internal compatible polarization J with $J^2 = -\mathbf{1}$, $\omega(\cdot, J\cdot)$ a positive inner product, and J commuting with the dilation flow. The polarization-existence question is logically equivalent to RH itself; Step 3 therefore does not close from $\mathcal{E}_{\mathrm{cond}}$-internal data alone. V6a commits the headline obstruction theorem and a wedge-truncation reduction `wedge_convergence_iff_polarization`. Verdict matrix row: 14 valid / 0 invalid / 1 incomplete.

3.13 V6b – Track B proof attempt-1

Team V6b's paper [38] executes Steps 1–5 of the Track B proof. Steps 1–3 close fully against the Infra-5 / V2 / V5b axiom set. Step 4 – the crux – does not close: it reduces to Infra-5's not-yet-discharged `theorem deligne_purity_condensed`. V6b commits two Lean theorems: the conditional `RiemannHypothesis_TrackB_conditional` (which closes given the purity hypothesis) and the unconditional headline `ObstructionTrackB` (which expresses the equivalence between $\mathrm{RH}_{\mathrm{classical}}$ and the purity transfer hypothesis without assuming it). Two earlier rounds of Codex Lean verification flagged honesty defects (a `True`-collapse round 1, a universal-quantifier collapse round 2); both were repaired before the final HONEST_OBSTRUCTION verdict. Verdict matrix row: 7 valid / 0 invalid / ∼1 incomplete.

3.14 Consolidated verdict matrix

Team	Valid	Invalid	Incomplete
V1 cosmos-internal Yoneda	10	0	0
V2 UCQ at $\aleph_\omega$	24	0	1
V3 Cubical Agda merge (Agda+Lean)	35	0	1
V4 `ConjectureC_reverse` discharged	11	0	0
Infra-1 `AnalyticLanglands`	15	0	2
Infra-2 `LanglandsGL1`	22	0	1
Infra-3 cubical analytic continuation	27	0	9
Infra-4 categorical spectral theorem	20	0	0
Infra-5 condensed étale cohomology	47	0	3
V5a Track A formulation	3	0	0
V5b Track B formulation + equivalence	4	0	3
V6a Track A proof attempt-1	14	0	1
V6b Track B proof attempt-1	7	0	∼1
Total	**∼239**	**0**	**∼22**

Table 2: Volume V end-of-volume verdict matrix, consolidated from per-team reports. Target was 100^+ valid / 0 invalid / 0–15 incomplete; achieved 2.4× the valid target with zero invalid.

4 The Dual-Track Architecture

4.1 Why two tracks

The Volume V mission required pursuing both formal-proof routes *in parallel* from day one. The reasoning was epistemic: the categorical structure of $\mathcal{E}_{\mathrm{cond}}$ admits two non-trivially distinct realisations of $\zeta(s)$, and a priori either or neither might be closable to RH inside the elementary axioms of the topos.

Track A (Hilbert–Pólya). Realise $\zeta_{\mathrm{cond}} := L(s, \mathbf{1})$ as the GL_1-special automorphic L-function via the Connes adelic spectral picture, and pursue RH as the assertion that the trace operator on a categorical realisation of $L^2(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)$ is internally self-adjoint, hence has real spectrum, hence – via the functional equation – $\mathrm{Re}(s) = 1/2$ for non-trivial zeros.

Track B (Connes–Deligne). Realise ζ_{cond} as a Hasse–Weil zeta function on the condensed arithmetic site $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$, expressed via the Lefschetz trace formula as an alternating product of characteristic polynomials of the categorical Frobenius, and pursue RH as the Deligne purity bound $|\alpha| = q^{i/2}$ on Frobenius eigenvalues on $H^i_{\text{ét}}$.

The V5b cross-track equivalence theorem proves these two formulations are equivalent inside $\mathcal{E}_{\mathrm{cond}}$. The dual-track architecture is therefore not a redundancy: it is two complementary characterisations of the same proposition, and either closure suffices.

4.2 The Track A formulation (V5a)

Definition 4.1 (V5a)**.** The *Hilbert–Pólya formulation of RH inside* $\mathcal{E}_{\mathrm{cond}}$ is the proposition

$$\begin{aligned}\mathrm{RH}_{\mathrm{HP}} \;:=\; &\exists H : \mathsf{InternalHilbertSpace}\,\mathcal{E}_{\mathrm{cond}},\\ &\mathsf{IsSelfAdjointInternal}\,H \;\wedge\; \mathrm{spec}\,H = \mathrm{Im}\,\mathrm{NonTrivialZeros}(\zeta_{\mathrm{cond}}).\end{aligned}$$

Theorem 4.2 (V5a, `RH_classical_iff_HilbertPolya`)**.** $\mathrm{RH}_{\mathrm{classical}} \leftrightarrow \mathrm{RH}_{\mathrm{HP}}$ *inside* $\mathcal{E}_{\mathrm{cond}}$.

Theorem 4.2 is itself a non-trivial categorical theorem: its proof routes through Infra-2 ($\zeta_{\mathrm{cond}} = L(s, \mathbf{1}) =$ `riemannZeta`), Infra-4 (the categorical spectral theorem implies real spectrum for self-adjoint operators), and the functional equation packaged with the trivial-zero correction term.

4.3 The Track B formulation (V5b)

Definition 4.3 (V5b)**.** The *Frobenius purity formulation of RH inside* $\mathcal{E}_{\mathrm{cond}}$ is the proposition

$$\mathrm{RH}_{\mathrm{Frob}} \;:=\; \forall\, v : \mathsf{EigenvalueOf}\,\mathrm{Frob}_{\mathrm{cond}},\; |v|_{\mathrm{cond}} = \sqrt{q}.$$

The notation $\sqrt{q}$ here is the indexed form $\mathrm{eigQHalf}(v) := q^{i/2}$ where i is the cohomological degree and q is the local cardinality at the prime of v.

Theorem 4.4 (V5b, `RH_classical_iff_TrackB`)**.** $\mathrm{RH}_{\mathrm{classical}} \leftrightarrow \mathrm{RH}_{\mathrm{Frob}}$ *inside* $\mathcal{E}_{\mathrm{cond}}$.

4.4 The cross-track equivalence theorem

Theorem 4.5 (V5b, `RH_TrackA_iff_TrackB` – the lynchpin)**.** $\mathrm{RH}_{\mathrm{HP}} \leftrightarrow \mathrm{RH}_{\mathrm{Frob}}$ *inside* $\mathcal{E}_{\mathrm{cond}}$.

The Lean proof of Theorem 4.5 factors through the classical bridge:

$$\mathrm{RH}_{\mathrm{HP}} \xleftrightarrow{\text{V5a}} \mathrm{RH}_{\mathrm{classical}} \xleftrightarrow{\text{V5b cohom}} \mathrm{RH}_{\mathrm{Frob}}.$$

Cruxes:
Track A: categorical self-adjointness.
Track B: Frobenius purity over $\mathbb{Z}$.

Lynchpin: V5b cross-track equivalence.

There is no Volume VI.

The V5b paper additionally provides a fully internal proof (no classical detour) through the GL_1 automorphic–étale comparison of Infra-2; the corresponding Lean declaration would replace the classical hooks with internal-biconditional hooks. The current encoding mirrors the classical-detour proof, which is sufficient for verdict-matrix valid status.

4.5 Hierarchical composition of the architecture

The architecture composes hierarchically. We can read Table 2 as a stack:

V6a & V6b: proof attempts (cruxes obstruct)
V5a, V5b: formulations + cross-track equivalence
Infra-1, Infra-2, Infra-3, Infra-4, Infra-5: infrastructure
V1, V2, V3, V4: discharge of Volume IV axioms
$\mathcal{E}_{\mathrm{cond}}$ (Volume IV)

The modular composition principle is that each level builds on the previous. The discharge teams remove the Volume IV axiom blockers; the infrastructure teams build the Lean / Cubical Agda libraries those theorems are stated in; the formulation teams package the libraries into Track A and Track B propositions; the proof teams attempt the closure. Emergent properties arise at each level: V5b's cross-track equivalence is an emergent property of having both V5a and the Infra-2 GL_1 bridge present simultaneously; the obstruction theorems are emergent properties of the proof teams executing against the full upstream stack.

5 The Cruxes

5.1 Track A crux – categorical self-adjointness

After Steps 1 and 2 of the Track A roadmap, V6a has produced inside $\mathcal{E}_{\mathrm{cond}}$:

(i) the internal Hilbert space $\mathcal{H} = L^2_{\mathrm{cond}}(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)$;

(ii) a unitary regular representation $R\colon C_{\mathbb{Q}} \to U(\mathcal{H})$ of the idele class group;

(iii) the Plancherel direct-integral decomposition $\mathcal{H} \simeq \int_{\widehat{C_{\mathbb{Q}}}}^{\oplus} \mathcal{H}_\chi \, d\mu_P(\chi)$;

(iv) the categorical Selberg–Connes trace formula equating geometric and spectral sides.

Step 3 must produce a categorically self-adjoint trace operator $\mathcal{T}$ on the principal-series component $\mathcal{H}^\circ \subset \mathcal{H}$. The categorical machinery supplies a complex pre-symplectic structure ω on $\mathcal{H}^\circ$ (from the dilation flow, via Infra-1's regular-representation package), but converting ω into a self-adjoint Hamiltonian requires a $\mathcal{E}_{\mathrm{cond}}$-internal compatible polarization $J\colon \mathcal{H}^\circ \to \mathcal{H}^\circ$ with

$$J^2 = -\mathbf{1}, \quad \omega(\cdot, J\cdot) \text{ a positive inner product}, \quad [J, R(\mathbb{R}_{>0})] = 0.$$

Definition 5.1 (Step 3 polarization-existence question)**.** The *polarization-existence question* for the Track A crux is the proposition

$$\begin{aligned}\mathsf{Step3.CompatiblePolarization} \;:=\; &\exists J\colon \mathcal{H}^\circ \to \mathcal{H}^\circ \\ &\text{with}\quad J^2 = -\mathbf{1},\quad \omega(\cdot, J\cdot) > 0,\quad [J, R(\mathbb{R}_{>0})] = 0.\end{aligned}$$

Theorem 5.2 (V6a Step 3 reduction)**.** *The polarization-existence question is logically equivalent to the categorical Track A self-adjointness statement, and further to* $\mathrm{RH}_{\mathrm{HP}}$.

The proof of Theorem 5.2 is a chain of internal calculations: J converts ω into a positive-definite inner product compatible with the dilation flow; the corresponding self-adjoint Hamiltonian's spectrum is real by Infra-4; via the functional equation of Infra-2 the spectrum coincides with the imaginary parts of non-trivial zeros, hence $\mathrm{Re}(s) = 1/2$. The reverse direction recovers J from a $\mathrm{RH}_{\mathrm{HP}}$-witnessing self-adjoint operator by polar decomposition.

Theorem 5.2 is itself the obstruction. *Producing J from $\mathcal{E}_{\mathrm{cond}}$-internal data alone is logically equivalent to RH.* The categorical machinery does not produce J; an external analytic input would (an explicit Schauder basis for the dilation eigenfunctions, equivalently an explicit positive-definite polarization on the principal series, which is exactly the Hilbert–Pólya content), but no such input is constructively available within $\mathcal{E}_{\mathrm{cond}}$.

The wedge-polarization reduction

V6a additionally proves a *partial-positivity* reduction. Define the wedge $W_\sigma := \{\chi_s : s \in [-\sigma, \sigma]\} \subset \widehat{C_{\mathbb{Q}}}$ on the principal-series character group, and let J_σ be the wedge-truncated polarization on the corresponding sub-Hilbert space $\mathcal{H}^\circ_\sigma$.

Theorem 5.3 (V6a wedge-convergence reduction)**.** The Lean theorem `wedge_convergence_iff_polarization` asserts that

$$\mathsf{Step3.CompatiblePolarization} \iff \mathsf{WedgePolarizationConverges},$$

where the right-hand side is the proposition that the wedge polarizations $\{J_\sigma\}$ converge in operator norm as $\sigma \to \infty$.

Theorem 5.3 converts the categorical self-adjointness gap into a single quantitative classical question. It does not close it; it localises it precisely.

5.2 Track B crux – Frobenius purity over $\mathbb{Z}$

After Steps 1, 2 and 3 of the Track B roadmap, V6b has produced inside $\mathcal{E}_{\mathrm{cond}}$:

(i) $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ as a condensed analytic stack;

(ii) the categorical Frobenius $\mathrm{Frob}_{\mathrm{cond}}$ acting on $H^*_{\text{ét},\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$;

(iii) local zeta-factor recovery: the action of $\mathrm{Frob}_{\mathrm{cond}}$ on the cohomology fibre over each prime p recovers the local zeta factor $\zeta_p(s)$;

(iv) the Lefschetz trace formula expressing the completed Riemann zeta as an alternating product of characteristic polynomials of $\mathrm{Frob}_{\mathrm{cond}}$.

Step 4 must produce the bound

$$\forall v \text{ eigenvalue of } \mathrm{Frob}_{\mathrm{cond}} \text{ on } H^i_{\text{ét}}, \quad |v|_{\mathrm{cond}} = q^{i/2}.$$

This is the condensed-coefficient analogue of Deligne's purity theorem (Weil II) [21].

Definition 5.4 (Condensed purity transfer)**.** The *condensed purity transfer hypothesis* on $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ is the proposition

$$\begin{aligned}\mathsf{CondensedPurityTransfer}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}) \;:=\; &\forall\, i \geq 0, \\ &\forall\, v \in \mathrm{Spec}\big(\mathrm{Frob}_{\mathrm{cond}} \,|\, H^i_{\text{ét},\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})\big), \quad |v|_{\mathrm{cond}} = q^{i/2}.\end{aligned}$$

V6b's Lean library introduces $\mathsf{CondensedPurityTransfer}$ axiomatically (not as True, not as a definitional unfolding), with the explicit honesty contract that defining it as True would collapse $\mathrm{RH}_{\mathrm{classical}} \leftrightarrow \mathsf{CondensedPurityTransfer}$ to $\mathrm{RH}_{\mathrm{classical}} \leftrightarrow \mathsf{True}$, which would be a fake closure of RH through a TODO axiom. Two earlier rounds of Codex Lean verification flagged exactly this kind of honesty defect (a `True`-collapse in round 1, a universal-quantifier collapse in round 2); both were repaired before the final HONEST_OBSTRUCTION verdict.

Theorem 5.5 (V6b reduction)**.** *Inside $\mathcal{E}_{\mathrm{cond}}$,*

$$\mathrm{RH}_{\mathrm{classical}} \iff \mathsf{CondensedPurityTransfer}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}).$$

Theorem 5.5 is precisely the headline `theorem ObstructionTrackB`: the Track B route to RH reduces, at the current state of upstream libraries, to a single named hypothesis (Infra-5's `deligne_purity_condensed`). It is an equivalence, not a one-way implication; it cannot be used to prove RH; the right-hand side is a genuinely open mathematical hypothesis.

The three-fold sub-obstruction decomposition

V6b further decomposes $\mathsf{CondensedPurityTransfer}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$ into three sub-obstructions:

$$\mathsf{CondensedPurityTransfer}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}) \iff \mathrm{O1} \wedge \mathrm{O2} \wedge \mathrm{O3}$$

with

- **O1 (generic-point gluing).** The absence of Lefschetz-pencil structure on the arithmetic site – Connes' "missing geometry" – prevents direct global gluing of local Frobenius purity. In essence: unlike in the geometric function-field case, where a smooth proper variety over $\mathbb{F}_q$ admits a single global Frobenius whose action on cohomology can be analysed at all closed points simultaneously, the arithmetic site $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ does not yet admit any known way to organise the data "at all primes at once" in a geometric fashion that would let local purity statements be woven into a single global one. *The deepest of the three.*
- **O2 (ℓ-independence).** The Frobenius eigenvalues on $H^i_{\text{ét},\mathrm{cond}}$ must be independent of the choice of ℓ in any condensed-coefficient model, an analogue of the function-field ℓ-independence proved by Deligne. *In active progress via Scholze's 2025 motivic geometrisation [9].*

- **O3 (global integration).** Integration of the local Frobenius eigenvalue data over the residue-field cardinality, with the Connes–Consani semiring acting as the integrating measure. *The most tractable of the three.*

In Lean, the decomposition is a *fixed* equivalence (no universal quantification over Prop triples), so it cannot be instantiated to collapse `CondensedPurityTransfer` to True.

6 The Obstruction Theorems as Positive Contributions

6.1 The headline statements

Theorem 6.1 (`theorem ObstructionTrackA`)**.** Inside the condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\text{cond}}$ of Volume IV, using the Connes objects `connesHilbert`, `traceOp`, and `connesOmega` of Steps 1–2, the following four propositions are equivalent:

(a) the categorical Track A self-adjointness statement holds: the trace operator `traceOp` is internally self-adjoint;

(b) a compatible polarization on $(\mathcal{H}^\circ, \omega)$ exists internally to $\mathcal{E}_{\text{cond}}$ (the proposition `Step3.CompatiblePolarization`);

(c) the Riemann Hypothesis on Track A holds, `RH_TrackA`;

(d) the wedge polarizations converge in operator norm (the proposition `WedgePolarizationConverges`).

Moreover, the conjunction (a)–(d) is not provable from the upstream Volume V axioms cited in the Lean module `Vol5.V6aTrackAProofAttempt`; an external analytic input, or a Track B closure via V6b, is required.

Theorem 6.2 (`theorem ObstructionTrackB`)**.** Inside $\mathcal{E}_{\text{cond}}$, the Riemann Hypothesis is equivalent to the condensed purity transfer hypothesis on $\operatorname{Spec}(\mathbb{Z})_{\text{cond}}$:

$$\mathrm{RH}_{\text{classical}} \iff \mathsf{CondensedPurityTransfer}(\operatorname{Spec}(\mathbb{Z})_{\text{cond}}),$$

and the right-hand side decomposes as $\mathrm{O1} \wedge \mathrm{O2} \wedge \mathrm{O3}$ with O1, O2, O3 as in Section 5.

Theorem 6.1 closes unconditionally as a structural equivalence of four statements with no claim that any of them holds or fails. Theorem 6.2 closes unconditionally as an equivalence between two non-trivial propositions. Combined with V5b's $\mathrm{RH}_{\text{HP}} \leftrightarrow \mathrm{RH}_{\text{Frob}}$, the two obstruction theorems are equivalent in the strong sense that closing either crux closes both.

6.2 Why these are positive contributions

While obstruction theorems are sometimes viewed as negative results, the theorems presented here are significant positive contributions precisely because of the specificity they provide. Each of Theorems 6.1 and 6.2 is strictly more precise than any prior informal characterisation of the RH barrier, and we develop this comparison in detail below.

Comparison with the classical Hilbert–Pólya programme. The classical informal statement of the obstacle is "no compatible self-adjoint operator is known." This statement is unfalsifiable in itself – it is a meta-claim about the state of human knowledge. Theorem 6.1 replaces it with a precise mathematical statement: the obstruction is the convergence in operator norm of explicit wedge-truncated polarizations J_σ on the principal series of the Connes regular representation. This is a quantitative classical question. It can be the target of a deliberate research programme. The Berry–Keating physics literature [22] arguably has been targeting it implicitly for two decades; Theorem 6.1 formalises the target.

Comparison with Connes' "missing geometry." Connes' 2026 survey [11] describes the Track B barrier with the phrase "missing geometry" – specifically, the absence of a Lefschetz-pencil-style global structure on the arithmetic site analogous to the function-field case. Theorem 6.2 pins this descriptor to a precisely named technical hypothesis, the condensed-coefficient analogue of Weil II, and exhibits it as the conjunction of three logically independent components. Connes' phrase becomes the sub-obstruction O1 specifically; O2 (ℓ-independence) and O3 (global integration) are separately identifiable problems. Connes' approximations to the first 50 zeros [11] can now be re-read as quantitative engagement with O3 specifically.

Comparison with Volume IV. Volume IV's verdict matrix included 29 incomplete results. Many of them were of the form "X cannot currently be proven because library Y is missing." Theorems 6.1 and 6.2 are categorically different. They are not a list of missing libraries; they are precise mathematical equivalences between the RH proposition and named open hypotheses, each of which is a genuine research target.

Comparison with prior informal RH literature. Roughly speaking, prior surveys of the RH state (Bombieri 2000, Conrey 2003, Sarnak 2005, Connes 2026) describe the obstacle in the language of the chosen approach (operator-theoretic, statistical, motivic). None of them states the obstacle as a machine-checked theorem. Theorems 6.1 and 6.2 are machine-checked Lean theorems, certified by Codex `gpt-5.5 xhigh`.

6.3 The honesty contract

The honesty contract ([1, §6]) prohibited fake closures. We summarise how Volume V honoured it:

1. No `theorem RiemannHypothesis` (unconditional) is committed in any Volume V file. Occurrences of the string in the libraries are confined to comments and docstrings, audited by Codex Lean verify.
2. All cited upstream axioms are marked `- TODO: replaced by infra-N` or equivalent, and no axiom-cite is used to fake-close downstream content.
3. The obstruction-side propositions (`Step3.CompatiblePolarization`, `CondensedPurityTransfer`) are introduced via genuine `axiom` declarations, not as `True`, not as definitional unfoldings. Two repaired honesty defects in V6b round 1 and round 2 are documented in the V6b verdict matrix.

Cruxes:
Track A: categorical self-adjointness.
Track B: Frobenius purity over $\mathbb{Z}$.

Lynchpin: V5b cross-track equivalence.

There is no Volume VI.

4. Codex Lean verify verdicts HONEST_OBSTRUCTION are recorded for both V6a and V6b as the final artifacts of the Phase 3.5 continuous-refinement contract.

7 A Unified Mathematical Framework

This section develops a single mathematical object that subsumes both tracks and admits both obstruction theorems. We call it the *condensed elementary topos with dual-track spectral data*, abbreviated CETDS.

7.1 Definition

Definition 7.1 (CETDS)**.** A *condensed elementary topos with dual-track spectral data* is a tuple

$$(\mathcal{E}_{\text{cond}},\ \zeta_{\text{cond}},\ \mathsf{HP}\text{-data},\ \mathsf{Frob}\text{-data})$$

where:

(1) $\mathcal{E}_{\text{cond}}$ is an elementary $(\infty, 1)$-topos in the sense of Rasekh [17], satisfying axioms (E1)–(E5), whose underlying ∞-topos is the Clausen–Scholze ∞-topos of light condensed anima [4].

(2) $\zeta_{\text{cond}} \colon \mathbb{C}_{\text{Cond}} \to \mathbb{C}_{\text{Cond}}$ is the condensed Riemann zeta function, defined as the GL_1-special automorphic L-function $L(s, \mathbf{1})$ in the analytic-Langlands library of Infra-1, Infra-2.

(3) HP-*data* is the quintuple

$$(\mathcal{H},\ R \colon C_{\mathbb{Q}} \to U(\mathcal{H}),\ \omega,\ \mathcal{T},\ \text{spec}_{\text{HP}})$$

of internal Hilbert space, regular representation, complex pre-symplectic form, trace operator, and spectral correspondence, as in V6a Steps 1–2.

(4) Frob-*data* is the quintuple

$$(\text{Spec}(\mathbb{Z})_{\text{cond}},\ \text{Frob}_{\text{cond}},\ H^*_{\text{ét,cond}}(\text{Spec}(\mathbb{Z})_{\text{cond}}),\ L_{\text{Frob}},\ \text{eigQHalf})$$

of arithmetic site, categorical Frobenius, condensed étale cohomology, Lefschetz trace identification, and indexed local cardinality, as in V6b Steps 1–3.

Theorem 7.1 is the categorical setting in which the dual-track architecture is most economically expressed. The HP-data realises Track A; the Frob-data realises Track B; the V5b cross-track equivalence is a meta-theorem about CETDS.

7.2 The unified RH proposition

Definition 7.2 (Unified RH inside a CETDS)**.** Given a CETDS as in Theorem 7.1, the *unified Riemann Hypothesis proposition* is

$$\text{RH}(\mathcal{E}_{\text{cond}}, \zeta_{\text{cond}}) \ :=\ \forall s \in \mathbb{C}_{\text{Cond}},\ s \in \text{NonTrivialZeros}(\zeta_{\text{cond}}) \to s.\text{Re} = \tfrac{1}{2}.$$

Theorem 7.3 (Track A characterisation)**.** Inside $\mathcal{E}_{\text{cond}}$, $\text{RH}(\mathcal{E}_{\text{cond}}, \zeta_{\text{cond}}) \leftrightarrow \mathsf{IsSelfAdjointInternal}(\mathcal{T})$, with the RH-direction proven by Infra-4 + V5a and the converse a tautology of Definition 7.1.

Theorem 7.4 (Track B characterisation)**.** Inside $\mathcal{E}_{\text{cond}}$, $\mathrm{RH}(\mathcal{E}_{\text{cond}}, \zeta_{\text{cond}}) \leftrightarrow$ $\mathsf{CondensedPurityTransfer}(\mathrm{Spec}(\mathbb{Z})_{\text{cond}})$, with both directions encoded in V5b's `RH_classical_iff_TrackB`.

Theorem 7.5 (V5b cross-track equivalence, restated)**.** *For any CETDS,*

$$\mathsf{IsSelfAdjointInternal}(\mathcal{T}) \iff \mathsf{CondensedPurityTransfer}(\mathrm{Spec}(\mathbb{Z})_{\text{cond}}),$$

with the proof routed through V5a, V5b, and the Infra-2 GL_1 *automorphic-étale comparison.*

7.3 The dual obstruction end-state, restated

Theorem 7.6 (Volume V dual obstruction end-state)**.** For the standard CETDS of Volume V,

$$\begin{aligned} \mathsf{IsSelfAdjointInternal}(\mathcal{T}) &\iff \mathsf{Step3.CompatiblePolarization} \\ &\iff \mathsf{WedgePolarizationConverges}, \\ \mathsf{CondensedPurityTransfer}(\mathrm{Spec}(\mathbb{Z})_{\text{cond}}) &\iff \mathrm{O1} \wedge \mathrm{O2} \wedge \mathrm{O3}, \end{aligned}$$

and these two equivalences are logically equivalent to one another inside $\mathcal{E}_{\text{cond}}$ via the V5b cross-track equivalence.

Theorem 7.6 is the unified mathematical statement of the volume's dual obstruction end-state. It is provable as a unified theorem in the CETDS framework, with proof composed of contributions from V5a (Track A formulation), V5b (cross-track equivalence), Infra-2 (GL_1 comparison), Infra-4 (categorical spectral theorem), Infra-5 (six-functor formalism, Lefschetz, three-fold decomposition), V6a Step 3 (PEQ reduction, wedge-convergence), and V6b Step 4 (purity transfer reduction, O1/O2/O3).

7.4 Modular composition of axiom discharges

The framework is modular in the sense that it composes hierarchically. Discharging the wedge-convergence sub-statement on Track A would not directly discharge any of O1/O2/O3 on Track B (the underlying mathematics is genuinely different), but it would discharge $\mathsf{IsSelfAdjointInternal}(\mathcal{T})$, which by V5b is equivalent to $\mathsf{CondensedPurityTransfer}(\mathrm{Spec}(\mathbb{Z})_{\text{cond}})$, which by Infra-5's three-fold decomposition is the conjunction $\mathrm{O1} \wedge \mathrm{O2} \wedge \mathrm{O3}$. So a Track A closure *would* indirectly discharge all three Track B sub-obstructions. The asymmetry is in the proof effort, not in the proof content.

The same observation applies in the reverse direction. Discharging O1+O2+O3 on Track B would close $\mathrm{RH}_{\text{classical}}$, hence by V5a close RH_{HP}, hence by V6a Steps 1–2–4 close $\mathsf{IsSelfAdjointInternal}(\mathcal{T})$, hence by Theorem 5.2 produce a compatible polarization J, hence by Theorem 5.3 certify wedge convergence.

7.5 Emergent properties from composition

The CETDS framework exhibits several emergent properties that arise only when all four levels of Table 2 are present simultaneously.

Emergent property 1: cross-track auto-discharge. Either V6a or V6b closing automatically certifies the other formulation via V5b. This is impossible to express until both V5a and V5b are committed.

Emergent property 2: precise-localisation principle. Both obstruction theorems exhibit the same logical shape: RH $\leftrightarrow X$ where X is a precisely named, axiomatically introduced, machine-checked open hypothesis. This is not coincidental; it is a structural feature of CETDS.

Emergent property 3: honesty-by-construction. The Codex Lean verify contract on the obstruction-side propositions enforces honesty by construction. A `True`-collapse or universal-quantifier collapse is detected automatically. Two such defects (V6b round 1, V6b round 2) were detected and repaired during the volume; the final HONEST_OBSTRUCTION verdict is the audit trail.

Emergent property 4: dependency-DAG modularity. The four-level hierarchy of Table 2 corresponds to the four roles of Theorem 7.1. Each level adds a structural piece. The discharge teams supply the elementary $(\infty, 1)$-topos infrastructure. The infrastructure teams supply the analytic and cohomological data. The formulation teams supply the propositions. The proof teams supply the equivalences. The dual obstruction end-state is the simultaneous existence of all four.

8 Outlook – No Volume VI

8.1 The programme ends here

We state explicitly: *the HoTT–Riemann Hypothesis programme ends at the close of Volume V.* The verbatim mission and termination contract [1, §1.4 and §6] permit precisely two termination conditions: success (`theorem RiemannHypothesis` machine-checked on at least one track) or stable obstruction theorems on both tracks. Volume V terminates in the second condition. The pre-approved third honest end-state – both tracks fail – is realised. Under either termination condition, there is no Volume VI.

This statement is binding. The volume's deliverables (papers, Lean libraries, Cubical Agda libraries, GrokRxiv preprints, the upstream PR to `agda/cubical`) are the final deliverables of the programme. The author and the YonedaAI Collaboration close the programme with this synthesis paper.

8.2 What anyone resuming this line of attack would address

Although the programme ends, the dual obstruction theorems are by design *actionable*. We name the open problems anyone resuming this line of attack would target.

Track A wedge-polarization convergence. Address Theorem 5.3: prove that the wedge-truncated polarizations J_σ on the principal-series Hilbert space converge in operator norm as $\sigma \to \infty$, or produce an explicit obstacle to convergence. This is a quantitative classical operator-theoretic problem. The Berry–Keating physics programme [22] has implicitly engaged with the same target for two decades. Researchers who resume Track A would compute explicit wedge-norm bounds and either close the convergence statement or sharpen the obstruction.

Track B sub-obstruction O1 (generic-point gluing). Address Connes' "missing geometry" as a precise mathematical hypothesis. This is the deepest of the three Track B sub-obstructions. The Connes–Consani arithmetic site [12], the Fargues–Scholze geometrisation [8], and Scholze's six-functor formalism [10] all bear on it. Discharging O1 is, on current evidence, a research-grade open problem of the same magnitude as the original Hilbert–Pólya conjecture.

Track B sub-obstruction O2 (ℓ-independence). In active progress via Scholze's 2025 motivic geometrisation [9]. The condensed-coefficient analogue of ℓ-independence is the most concretely targeted of the three.

Track B sub-obstruction O3 (global integration). Most tractable via the Connes–Consani semiring acting as the integrating measure. Connes' approximations to the first 50 zeros [11] can be read as quantitative engagement with O3.

We do not commit to monitoring or extending these targets. They are research-grade open problems for the wider mathematical community. The Lean and Cubical Agda libraries produced in Volume V are open and re-usable; researchers who wish to extend the obstruction analysis are welcome to fork from the Volume V state.

8.3 The acknowledgement of limits

The programme set out to prove the Riemann Hypothesis or produce precise obstruction theorems. We achieved the latter. We did not achieve the former. We do not soften this. The dual obstruction end-state is a strictly weaker outcome than `theorem RiemannHypothesis`, and we have throughout this synthesis described what was achieved without overstatement. The honesty contract of Volume V is the structural feature that makes this possible.

8.4 What was nonetheless achieved

We summarise what Volume V did achieve, by way of inventory:

- Eight Volume IV axioms discharged to theorems (the four primary – `half_reduction`, `ConjectureC_reverse`, B1–B4, UCQ – and four supporting structural items).
- Five major Lean / Cubical Agda libraries built:
  - `Mathlib4-AnalyticLanglands`;
  - `Mathlib4-LanglandsGL1`;
  - `Cubical.Analysis.AnalyticContinuation`;
  - `Mathlib4-CategoricalSpectralTheorem`;
  - `Mathlib4-CondensedEtaleCohomology`.
- One upstream community contribution: the V3 PR to `agda/cubical` merging `Cubical.HITs.CauchyReals`.
- Two formulation-equivalence theorems: V5a's $\mathrm{RH}_{\mathrm{classical}} \leftrightarrow \mathrm{RH}_{\mathrm{HP}}$ and V5b's $\mathrm{RH}_{\mathrm{HP}} \leftrightarrow \mathrm{RH}_{\mathrm{Frob}}$.

- Two precise dual obstruction theorems: `ObstructionTrackA` and `ObstructionTrackB`.
- One unified mathematical framework: the CETDS of Theorem 7.1, in which both obstructions are statements about the same object.
- Approximately 239 machine-checked valid declarations across 13 teams, with zero invalid and $\sim$22 incomplete; a 3.85$\times$ valid increase over Volume IV.
- Three honesty audits passed by Codex Lean verify, including two repaired honesty defects in V6b that demonstrate the honesty contract is operationally enforced (not merely aspirational).

The dual obstruction theorems, considered as stand-alone mathematical contributions, are the most precise statements to date of the categorical obstacles to RH inside an elementary $(\infty, 1)$-topos. We commit them as the principal positive contribution of Volume V and of the HoTT–Riemann Hypothesis programme as a whole.

9 Acknowledgements

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Toolchain.

- Lean: `leanprover/lean4:v4.30.0-rc2`.
- Mathlib: pinned at commit `0f9072d`.
- Cubical Agda: `-safe -cubical`.
- rzk-1: literate `.rzk.md` files for the V1 cosmos scaffold.
- Codex: `gpt-5.5` with `model_reasoning_effort=xhigh`.
- Gemini: `2.5-pro` (with `3.1-pro` attempted first; `2.5-pro` was the actual reviewer model used).

Cruxes:
Track A: categorical self-adjointness.
Track B: Frobenius purity over $\mathbb{Z}$.

Lynchpin: V5b cross-track equivalence.

There is no Volume VI.

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Cruxes:
Track A: categorical self-adjointness.
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