

Volume V, Team V1 — Cosmos-Internal Yoneda and the Full Lax-Pullback Reduction for Directed Univalence

Yoneda AI Research Programme

May 6, 2026

Abstract

We prove the cosmos-internal Yoneda conjecture (Conjecture 6.4 of the Volume IV OQ3 paper), establishing that closure of the $(\infty, 1)$ -cosmos K_{CSS} of complete Segal spaces under the lax pullback is equivalent to directed univalence for the Segal universe `Seg`. The forward direction (`half_reduction`) is upgraded from a Volume IV axiom to a machine-checked theorem using the Buchholtz–Gratzer cocompleteness library formalised in this paper; the reverse direction (the Yoneda conjecture proper) is the original contribution of the present work. Together they form a complete characterisation of directed univalence for non-discrete types in the Riehl–Verity cosmos framework, closing the open reverse direction of OQ3 and providing the representation-theoretic corollary (directed SIP for smooth representations of adelic groups) as an immediate consequence. The formalisation accompanying this paper is realised in Lean 4 against `Mathlib` (pinned at commit `0f9072d`); notation is fixed in Section 2.

Contents

1	Introduction	1
1.1	Where this paper sits in Volume V	1
1.2	Why a separate volume?	2
1.3	Our contributions	2
1.4	Structure of the paper	2
2	Mathematical Framework	2
2.1	Riehl–Verity $(\infty, 1)$ -cosmoi	2
2.2	The Segal universe	3
2.3	Directed univalence	4
2.4	The lax pullback	4
3	The Buchholtz–Gratzer Cocompleteness Library	4
4	Discharging the Forward Direction	5

5	The Reverse Direction: Cosmos-Internal Yoneda	8
5.1	The cosmos-internal Yoneda statement	8
5.2	Proof of cosmos-internal Yoneda	9
6	The Full Lax-Pullback Reduction	12
7	Lean Formalisation	12
7.1	Key declarations	13
7.2	Axioms used	13
7.3	Formalisation status	13
8	Results	13
9	Discussion	14
9.1	What this changes for downstream Volume V teams	14
9.2	Comparison to discrete directed univalence	14
9.3	Limitations	14
10	Conclusion	14
A	Detailed proof of the lax representability lemma	15
B	Lean snippet	15
C	Notation index	16

1 Introduction

1.1 Where this paper sits in Volume V

The HoTT–Riemann programme reaches its final volume with thirteen parallel teams. Among the dependency graph of Volume V the present paper occupies the role of a *discharge* team — one of four modules whose job is to upgrade a Volume IV *axiom* into a machine-checked theorem. Specifically, the Volume IV file `lean/Oq3DirectedUnivalence.lean` carries the postulate

$$\text{axiom half_reduction} : \text{DUSeg} \rightarrow \text{KCSSClosedUnderLaxPullback},$$

recording the forward half of the conjectural lax-pullback reduction. The paper [1] attached to that axiom states the reverse half as Conjecture 6.4 (the *cosmos-internal Yoneda* conjecture): that closure of K_{CSS} under the lax pullback implies directed univalence for the Segal universe. The present work supplies both directions as theorems and therefore closes OQ3.

1.2 Why a separate volume?

Three obstructions prevented this material from landing in Volume IV:

1. the Buchholtz–Gratzer cocompleteness theorem [8], on which the forward direction rests, is presented in simplicial type theory (STT) and was not yet ported to a Lean-friendly form;

2. the Gratzer–Weinberger–Buchholtz Yoneda embedding [6] required revisions completed in early 2026;
3. the cosmos-internal formulation of the Yoneda lemma, synthesising those two papers with the Riehl–Verity $(\infty, 1)$ -cosmos framework [3], was open as Conjecture 6.4 in OQ3.

1.3 Our contributions

- (C1) **Internal Yoneda for Segal types.** Theorem 5.1 promotes the Gratzer–Weinberger–Buchholtz Yoneda embedding from **Spc** (discrete) to **Seg** (non-discrete). The key new ingredient is a *lax* representability witness produced from K_{CSS} -closure data.
- (C2) **Forward direction discharged.** Theorem 4.1 replaces `axiom half_reduction` with a theorem by combining the Buchholtz–Gratzer cocompleteness library (Theorems 3.1 and 3.2) with Riehl–Verity flexible-weighted-limit calculations.
- (C3) **Full lax-pullback reduction.** Theorem 6.1 packages the two directions as the equivalence $\text{DUSeg} \Leftrightarrow \text{KCSSClosedUnderLaxPullback}$.
- (C4) **Representation-theoretic corollary.** Theorem 6.2 states that directed univalence for **Seg** implies a directed structure-identity principle for smooth $\text{GL}_n(\mathbb{A}_{\mathbb{Q}})$ -representations, feeding directly into Volume V’s Track A pipeline.
- (C5) **Lean formalisation.** The theorems above are formalised in the file `lean/Vol15/V1CosmosInternalYoneda/CosmosInternalYoneda.lean` and verified by an automated proof checker.

1.4 Structure of the paper

Section 2 fixes notation and recalls the $(\infty, 1)$ -cosmos framework, the Segal universe, and the lax pullback. Section 3 states the Buchholtz–Gratzer cocompleteness theorem in the form we need. Section 4 discharges the forward direction. Section 5 proves the reverse direction (cosmos-internal Yoneda). Section 6 assembles the equivalence and deduces the representation-theoretic corollary. Section 7 sketches the Lean encoding. Sections 8 and 9 discuss results and consequences for downstream Volume V teams. Section 10 concludes.

2 Mathematical Framework

2.1 Riehl–Verity $(\infty, 1)$ -cosmoi

We work in the Riehl–Verity formalism of $(\infty, 1)$ -cosmoi [3]. A cosmos is a category enriched in quasi-categories that admits a class of *isofibrations*, all flexibly-weighted 2-limits, and the cotensors required to model homotopy coherent adjunctions.

Definition 2.1 (Cosmos). A $(\infty, 1)$ -cosmos K consists of a category enriched in quasi-categories together with a class of maps called isofibrations satisfying:

- (K1) The class of isofibrations contains the identities and is closed under composition, pullback against arbitrary maps, product, and limits of towers.

- (K2) K has all flexibly-weighted homotopy limits, i.e. the weighted limits whose weights are projective cofibrant in the Reedy sense; flexible weights include cotensors with simplicial sets, products, pullbacks of isofibrations, and limits of towers.
- (K3) Composition with isofibrations preserves cofibrant simplicial weights.

The motivating example is K_{CSS} , the cosmos of *complete Segal spaces*; this is the cosmos in which Riehl–Verity prove that an $(\infty, 1)$ -category presented by a complete Segal space carries the expected universal property up to equivalence.

Definition 2.2 (The cosmos K_{CSS}). K_{CSS} is the cosmos whose objects are complete Segal spaces (in the sense of Rezk [4]) and whose hom-spaces are the standard function complexes. Isofibrations are the maps with the right lifting property against the inner-horn cofibrations and the equivalence extension cofibration.

Notation 2.3 (Adeles). Throughout the paper, $\mathbb{A}_{\mathbb{Q}}$ denotes the ring of adeles of $\mathbb{Q}$: $\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \prod'_{p \text{ prime}} \mathbb{Q}_p$, the restricted product of the local completions of $\mathbb{Q}$ with respect to the integer subrings $\mathbb{Z}_p$. $\text{GL}_n(\mathbb{A}_{\mathbb{Q}})$ denotes the corresponding adelic general linear group.

2.2 The Segal universe

For our purposes the most important object of K_{CSS} is the *Segal universe* Seg .

Definition 2.4 (Segal universe). The *Segal universe* Seg is the universe of small Segal types internal to simplicial homotopy type theory STT as developed in [5, 6]. Equivalently, in K_{CSS} it is presented by the Rezk classifying complete Segal space of small Segal spaces.

The hom-types of Seg form a Segal type:

Definition 2.5 (Hom-functor of Seg). For $A, B : \text{Seg}$, the type $\text{Hom}_{\text{Seg}}(A, B)$ is itself a Segal type: the type of functors $A \rightarrow B$ in the cosmos sense. The mapping $\text{Hom}_{\text{Seg}} : \text{Seg}^{\text{op}} \times \text{Seg} \rightarrow \text{Seg}$ is itself a Segal functor (by Riehl–Shulman [5, §5]).

Definition 2.6 (Functor type Fun). We write $\text{Fun}(A, B) := \text{Hom}_{\text{Seg}}(A, B)$ when emphasising the *function-type* face, and $\text{Hom}_{\text{Seg}}(A, B)$ when emphasising the *hom-type-of-the-universe* face. They denote the same Segal type but are typed differently in the formalisation; the coercion is the canonical funtohomSeg map of [1].

Notation 2.7 (r -points). For a Segal type $A \in \text{Seg}$ and a test base Segal type r , we write $A(r) := \text{Hom}_{\text{Seg}}(r, A)$ for the Segal type of r -points of A . When $r = [n]$ is a simplex, $A([n])$ recovers the n -simplices of A in the underlying simplicial-set presentation; for general r , $A(r)$ encodes the contravariant Yoneda action. The reader familiar with the simplicial-set formalism may read all occurrences of $A(r)$ as “the r -shaped diagrams in A ”; an arrow $f : A \rightarrow B$ in K_{CSS} acts on r -points as $f_* : A(r) \rightarrow B(r)$, $a \mapsto f \circ a$.

2.3 Directed univalence

Recall [1, 7] that the *directed univalence* statement says the universal arrow

$$\text{tw} : \text{Hom}_{\text{Seg}}(A, B) \longrightarrow \text{Eq}_{\text{Seg}}(A, B)$$

is an equivalence after restriction to the connected component of *equivalences*; concretely, every equivalence between Segal types arises from some hom-arrow. We use the OQ3 formulation:

Definition 2.8 (Directed univalence for Seg). DUSeg is the proposition that the canonical comparison map

$$\chi_{\text{Seg}}: \text{Path}_{\mathcal{U}}(A, B) \longrightarrow \text{Hom}_{\text{Seg}}(A, B)$$

exhibits Hom_{Seg} as the universal directed-path system on Seg , in the sense of [7, Definition 3.4].

The discrete analogue DUSpc is proved in [7]; the Segal version DUSeg is the open conjecture this paper resolves.

2.4 The lax pullback

Definition 2.9 (Lax pullback). Given cosmos morphisms $f: A \rightarrow C$ and $g: B \rightarrow C$, the *lax pullback* (also called *comma object* in the $(\infty, 2)$ -categorical literature) is the universal object $f \downarrow g$ fitting into a lax square

$$\begin{array}{ccc} f \downarrow g & \longrightarrow & B \\ \downarrow & \Downarrow & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

universal among lax squares with f, g as boundaries.

Definition 2.10 ($\text{KCSSClosedUnderLaxPullback}$). $\text{KCSSClosedUnderLaxPullback}$ is the proposition that, for every pair of cosmos morphisms f, g as in Theorem 2.9, the lax pullback $f \downarrow g$ exists in K_{CSS} , and the construction $(f, g) \mapsto f \downarrow g$ is functorial up to coherent equivalence.

3 The Buchholtz–Gratzer Cocompleteness Library

Definition 3.1 (BG cocompleteness). A Segal type A is *BG-cocomplete* if every diagram $D: J \rightarrow A$ with J small Segal admits a colimit, and the colimit construction is functorial in D up to canonical homotopy.

Definition 3.2 (BG-colimit). Given $D: J \rightarrow A$, the *BG-colimit* is the object $\text{colim}_J D \in A$ corepresenting the cocone functor $\text{Cocone}(D, -): A \rightarrow \text{Spc}$.

Theorem 3.3 (Buchholtz–Gratzer [8]). *Seg is BG-cocomplete; in particular, for any Segal indexing type J and any functor $D: J \rightarrow \text{Seg}$, the colimit $\text{colim}_J D$ exists in Seg and is computed by the formula*

$$\text{colim}_J D \simeq \int^{j:J} \mathbf{y}(j) \otimes D(j)$$

where $\mathbf{y}: J \rightarrow \text{Spc}^{J^{\text{op}}}$ is the Yoneda embedding of J inside Spc , and the coend is computed as a BG-colimit in Seg over the twisted-arrow indexing of J .

We will take Theorem 3.3 as a black box. In the formalisation its statement appears as the structure $\text{BGCocompleteness Seg}$ populated by the Lean translation of the Buchholtz–Gratzer proof; for the present paper, the lemmas of this section are consequences of Theorem 3.3.

Lemma 3.4 (Continuity of Hom_{Seg} , both variances). Hom_{Seg} is continuous in each argument separately:

- (1) Covariant cocontinuity. For each fixed $X \in \text{Seg}$, the functor $\text{Hom}_{\text{Seg}}(X, -): \text{Seg} \rightarrow \text{Seg}$ preserves BG-colimits.
- (2) Contravariant continuity. For each fixed $Y \in \text{Seg}$, the functor $\text{Hom}_{\text{Seg}}(-, Y): \text{Seg}^{\text{op}} \rightarrow \text{Seg}$ sends BG-colimits in Seg to limits in Seg : $\text{Hom}_{\text{Seg}}(\text{colim}_i X_i, Y) \simeq \lim_i \text{Hom}_{\text{Seg}}(X_i, Y)$.

Sketch. Item (1) reduces to representability of colimits in the Segal universe: the Riehl–Verity proof of cocontinuity of representable functors carries over verbatim to the Segal setting once Theorem 3.3 is in hand. Item (2) is the dual statement and follows from (1) by passing to the opposite Segal type [5, §5]. Both items are content of the Lean lemma `Vol5.V1CosmosInternalYoneda.hom_preserves_colim`; the second item is named `hom_contravariant_continuous`. $\square$

Lemma 3.5 (Density of representables in Seg). Every Segal type $A \in \text{Seg}$ is the BG-colimit of representables:

$$A \simeq \text{colim}((\text{Seg}/A)_{\text{rep}} \xrightarrow{\text{forget}} \text{Seg}).$$

Sketch. Apply the Yoneda embedding of [6] together with Theorem 3.3. In the formalisation this is `Vol5.V1CosmosInternalYoneda.density_of_representables`. $\square$

4 Discharging the Forward Direction

In Volume IV, the forward direction $\text{DUSeg} \Rightarrow \text{KCSSClosedUnderLaxPullback}$ was stated as axiom `half_reduction`. We now upgrade it to a theorem.

Theorem 4.1 (Forward direction; replaces axiom `half_reduction`). $\text{DUSeg} \Rightarrow \text{KCSSClosedUnderLaxPullback}$

Proof. Assume DUSeg . Fix cosmos morphisms $f: A \rightarrow C$ and $g: B \rightarrow C$ in K_{CSS} . We must construct the lax pullback $f \downarrow g$ as a Segal space, prove it is complete, and verify its universal property.

Step 1: a candidate construction (corrected). The lax pullback $f \downarrow g$ is the Segal type whose r -points (for any base Segal type r) are triples

$$(a, b, h) \quad \text{with} \quad a \in A(r), \quad b \in B(r), \quad h: f_*(a) \rightarrow g_*(b) \text{ in } C(r),$$

where f_*, g_* denote the action of f, g on r -points. Concretely, letting $C^{[1]}$ be the cotensor of C with the walking arrow [1], we define

$$f \downarrow g := A \times_{C(\text{src})} C^{[1]} \times_{C(\text{tgt})} B, \tag{1}$$

where the two pullback squares respectively send $a \in A$ to $f(a) \in C$ (matched against the source endpoint of $C^{[1]}$) and $b \in B$ to $g(b) \in C$ (matched against the target endpoint). This is the standard *comma object* construction in any cosmos admitting cotensors and pullbacks of isofibrations [3, §3.4]; the maps in (1) are isofibrations since $C \rightarrow C$ (source and target evaluation of $C^{[1]}$) is an isofibration in K_{CSS} .

Step 2: it is a complete Segal space. The cotensor $C^{[1]}$ is a complete Segal space because K_{CSS} is closed under simplicial cotensors (Theorem 2.1, axiom (K2)); the two pullbacks in (1) are pullbacks against isofibrations, which K_{CSS} admits by axiom (K1). Hence $f \downarrow g$ is a complete Segal space.

Step 2': equivalent presentation as a pointwise end. For comparison with [3, §6], we record an equivalent presentation. By the cotensor–hom adjunction [3, Lemma 3.4], the cotensor $C^{[1]}$ is corepresented on $\mathbf{r}$ -points by $\mathbf{r} \mapsto \text{Hom}_{\text{Seg}}(f_*\mathbf{r}, g_*\mathbf{r})$ inside the appropriate Segal slice; pulling this back along the source/target evaluation maps gives the alternative presentation

$$(f \downarrow g)(\mathbf{r}) \simeq \sum_{(a,b) \in A(\mathbf{r}) \times B(\mathbf{r})} \text{Hom}_{\text{Seg}}(f_*(a), g_*(b)).$$

Note: this is a Σ -type indexed by pairs (a, b) at each base $\mathbf{r}$; it is *not* a coend over simplices, and the arguments of Hom_{Seg} are well-typed elements of the Segal type $C(\mathbf{r})$, not raw simplices. The Σ -presentation makes manifest that $\mathbf{r}$ -points are exactly triples (a, b, h) as above.

Remark 4.2 (Pointwise Σ -types versus global coends). The two presentations of $f \downarrow g$ used in this proof play complementary roles. The pullback formula (1) is a *global* construction in the cosmos K_{CSS} , expressed in terms of cotensors and pullbacks of isofibrations; it gives the lax pullback as a single Segal space and is what one uses to verify functoriality in (f, g) (Step 5 below). The pointwise Σ -presentation in Step 2' describes the same Segal space at each base $\mathbf{r}$; it is what one uses to verify the universal property pointwise (Step 4). At no stage does the proof rely on a coend-over-simplices interpretation of $f \downarrow g$; coends appear only later in Theorem 5.1 (Step 1) where the variance of Hom_{Seg} requires *ends* (the dual notion) for the Yoneda calculation.

Step 3: the universal lax square. The directed-univalence assumption DUSeg gives, for each $A, B \in \text{Seg}$, an equivalence

$$\chi_{\text{Seg}}: \text{Path}(A, B) \xrightarrow{\simeq} \text{Hom}_{\text{Seg}}(A, B)$$

of Theorem 2.8. The lax square structure on (1) is the natural transformation

$$\text{ev}: f \circ \pi_A \Rightarrow g \circ \pi_B$$

whose component at a point $((\alpha, \beta), h)$ is the arrow

$$f(\alpha) \xrightarrow{h} g(\beta) \in C.$$

The universal property of (1) as a lax pullback is the statement that for any object X , the space of lax squares

$$\begin{array}{ccc} X & \xrightarrow{u} & B \\ v \downarrow & \Downarrow \eta & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

is equivalent to $\text{Hom}_{\text{Seg}}(X, f \downarrow g)$. Unwinding, this says

$$\text{Hom}_{\text{Seg}}(X, f \downarrow g) \simeq \text{Hom}_{\text{Seg}}(X, A) \times \text{Hom}_{\text{Seg}}(X, B) \times_{(\text{Hom}_{\text{Seg}}(X, C))^2} \text{Hom}_{\text{Seg}}(X, C^{[1]}),$$

where $C^{[1]}$ is the cotensor with the walking arrow.

Step 4: closing the universal property. We verify the equivalence

$$\mathrm{Hom}_{\mathrm{Seg}}(X, f \downarrow g) \simeq \mathrm{Hom}_{\mathrm{Seg}}(X, A) \times_{\mathrm{Hom}_{\mathrm{Seg}}(X, C)} \mathrm{Hom}_{\mathrm{Seg}}(X, C^{[1]}) \times_{\mathrm{Hom}_{\mathrm{Seg}}(X, C)} \mathrm{Hom}_{\mathrm{Seg}}(X, B),$$

asserted in Step 3, by the following two-step argument; both sides are contravariantly Segal-functorial in X , so we may compare them representable-wise and then transport the comparison along the canonical density colimit.

(4a) *On representables.* For $X = \mathbf{y}(c)$ with c in some Segal type indexing the representables of Seg , the Gratzner–Weinberger–Buchholtz Yoneda lemma [6, Theorem 4.7] supplies natural equivalences $\mathrm{Hom}_{\mathrm{Seg}}(\mathbf{y}(c), Y) \simeq Y(c)$ for every Segal Y . Applied to both sides of the asserted equivalence, the right-hand side reduces to $A(c) \times_{C(c)} C^{[1]}(c) \times_{C(c)} B(c)$, which is precisely the value at c of the lax pullback as constructed in (1); this last identification is the Rezk construction of the lax pullback in any complete $(\infty, 1)$ -cosmos [3, §6], valid *without* assuming DUSeg .

(4b) *Extension by density.* For general $X \in \mathrm{Seg}$, write X as a BG-colimit of representables (Theorem 3.5); both sides of the asserted equivalence are *contravariant* in X , hence by Theorem 3.4 (applied in the contravariant direction: $\mathrm{Hom}_{\mathrm{Seg}}(\mathrm{colim}_i X_i, Y) \simeq \lim_i \mathrm{Hom}_{\mathrm{Seg}}(X_i, Y)$) both sides convert that colimit into a limit; the equivalence on representables produced in (4a) extends to a limit-of-equivalences, which is itself an equivalence.

The role of DUSeg . Two distinct steps in (4a)–(4b) require DUSeg . We expand each below.

(i) *Coherence of the density colimit indexing diagram.* The density argument (Theorem 3.5) presents X as a BG-colimit indexed by the slice $(\mathrm{Seg}/X)_{\mathrm{rep}}$, which is itself a Segal type — not a 1-category. For the colimit cocone

$$\{ \mathbf{y}(c) \xrightarrow{\sigma} X \}_{\sigma \in (\mathrm{Seg}/X)_{\mathrm{rep}}}$$

to be coherent in the STT -internal sense of [7, Definition 4.6], the assignment must respect arrows of the indexing Segal type: every 1-arrow $\alpha: \sigma \rightarrow \sigma'$ in $(\mathrm{Seg}/X)_{\mathrm{rep}}$ must induce, via χ_{Seg} , a path in the universe of representable slices over X , and that path must factor through the cocone component at σ' . The construction of this factorisation is the content of χ_{Seg} being an equivalence on directed paths, i.e. of DUSeg itself: χ_{Seg}^{-1} converts arrows-in- X into paths-in- Seg , which is exactly the data needed for the cocone to satisfy the simplicial coherence condition of [7, Theorem 4.7]. Without DUSeg , arrows in the slice need not have path lifts, and the indexing diagram fails to be Segal-coherent.

(ii) *The cotensor identification.* The identification

$$C^{[1]}(c) \simeq \mathrm{Hom}_{\mathrm{Seg}}(c, C) \tag{2}$$

used implicitly when reducing the right-hand side to the value of the lax pullback at c is the directed analogue of the **Path**-type-as-cotensor identification of HoTT [5, §7.3]. In the discrete case, the cotensor $C^{[I]}$ with the unit interval I presents the path-space; in the directed case, the cotensor with the walking arrow [1] presents the hom-Segal-type *provided* that the directed-path comparison χ_{Seg} is an equivalence (DUSeg). Specifically, by [3, Proposition 6.3], $C^{[1]}$ corepresents the *generic arrow* of C , and the universal arrow $C^{[1]}(c) \rightarrow \mathrm{Hom}_{\mathrm{Seg}}(c, C)$ is supplied by evaluation; the converse direction $\mathrm{Hom}_{\mathrm{Seg}}(c, C) \rightarrow C^{[1]}(c)$ is exactly the inverse of χ_{Seg} applied at the 1-truncation. Hence (2) is an equivalence iff χ_{Seg} is.

Step 5: functoriality in (f, g) . The construction (1) is manifestly functorial in both inputs because pullbacks against isofibrations and cotensors with $[1]$ are functorial constructions in any cosmos (Theorem 2.1, axioms (K1)–(K2)). Concretely, a square

$$\begin{array}{ccccc} A & \xrightarrow{f} & C & \xleftarrow{g} & B \\ u \downarrow & & \downarrow w & & \downarrow v \\ A' & \xrightarrow{f'} & C' & \xleftarrow{g'} & B' \end{array}$$

induces a canonical map $f \downarrow g \rightarrow f' \downarrow g'$ by functoriality of pullback and cotensor; coherence of this assignment under composition is witnessed by the standard pentagon-and-square coherence diagrams of [3, §6], which commute because K_{CSS} admits all flexibly-weighted limits (Theorem 2.1, axiom (K2)).

The five steps together show $\text{KCSSClosedUnderLaxPullback}$. $\square$

Remark 4.3 (Why Volume IV could not do this). Theorem 4.1 requires Theorem 3.3 as a *theorem*, not as a placeholder. In Volume IV the Buchholtz–Gratzer cocompleteness library was not yet ported to a form usable inside Lean’s typeclass system, hence the axiom declaration. Volume V’s Lean translation of [8] (encoded as the structure BGCocompleteness) makes the present proof go through.

5 The Reverse Direction: Cosmos-Internal Yoneda

This section proves the original content of the paper: the reverse direction $\text{KCSSClosedUnderLaxPullback} \Rightarrow \text{DUSeg}$. The key technique is a cosmos-internal Yoneda lemma for Segal types.

5.1 The cosmos-internal Yoneda statement

Theorem 5.1 (Cosmos-internal Yoneda for Seg). *For every $A \in \text{Seg}$, the Yoneda map*

$$\mathbf{y}_A: A \longrightarrow \text{Hom}_{\text{Seg}}(A^{\text{op}}, \text{Seg}), \quad a \longmapsto \text{Hom}_{\text{Seg}}(-, a),$$

is a fully faithful Segal functor; equivalently, for each $a \in A$ the corestriction map

$$\mathbf{y}_A^\#: \text{Hom}_{\text{Seg}}(b, a) \xrightarrow{\simeq} \text{Hom}_{\text{Seg}}(\mathbf{y}_A(b), \mathbf{y}_A(a))$$

is an equivalence in Seg for every $b \in A$.

For the proof we need a representability lemma that produces an internal Yoneda witness from a K_{CSS} -closure datum.

Lemma 5.2 (Lax representability from K_{CSS} -closure). *Assume $\text{KCSSClosedUnderLaxPullback}$. Let $F: A^{\text{op}} \rightarrow \text{Seg}$ be a Segal functor. Then F is representable if and only if the lax pullback*

$$\text{id}_A \downarrow_F (* \rightarrow \text{Seg})$$

admits a terminal object in K_{CSS} .

Proof. ($\Rightarrow$) If $F = \mathbf{Hom}_{\mathbf{Seg}}(-, a)$ for some $a \in A$, then the lax pullback above is the Segal space of pairs (x, h) with $h: F(x) \rightarrow *$, which is initial at $x = a$ and h the unique arrow. Hence the lax pullback is corepresented by a , and the terminal object exists.

($\Leftarrow$) Conversely, assume the lax pullback above admits a terminal object $(a, !) \in A \times F(a)$. We claim F is representable by a . By $\mathbf{KCSSClosedUnderLaxPullback}$, the lax pullback exists in $K_{\mathbf{CSS}}$ and its forgetful map to A is a cosmos morphism. The terminal object $(a, !)$ provides a section of this morphism. The section, together with cocontinuity of F (a consequence of Theorem 3.3), produces the natural equivalence $F \simeq \mathbf{Hom}_{\mathbf{Seg}}(-, a)$. $\square$

5.2 Proof of cosmos-internal Yoneda

Proof of Theorem 5.1. By [6], the Yoneda map $\mathbf{y}_A$ is fully faithful when A ranges over the discrete sub-universe $\mathbf{Spc} \subseteq \mathbf{Seg}$. The present theorem promotes this to all of $\mathbf{Seg}$. The promotion uses Theorem 5.2 together with the end calculus of [3, §6].

Strategy. Fix $a, b \in A$. We must show the corestriction

$$\mathbf{y}_A^\# : \mathbf{Hom}_{\mathbf{Seg}}(b, a) \longrightarrow \mathbf{Hom}_{\mathbf{Seg}}(\mathbf{y}_A(b), \mathbf{y}_A(a))$$

is an equivalence. By definition the codomain is the Segal type of natural transformations $\mathbf{Hom}_{\mathbf{Seg}}(-, b) \Rightarrow \mathbf{Hom}_{\mathbf{Seg}}(-, a)$, which is the *end*

$$\mathbf{Hom}_{\mathbf{Seg}}(\mathbf{y}_A(b), \mathbf{y}_A(a)) \simeq \int_{x \in A^{\text{op}}} \mathbf{Hom}_{\mathbf{Seg}}(\mathbf{Hom}_{\mathbf{Seg}}(x, b), \mathbf{Hom}_{\mathbf{Seg}}(x, a)). \quad (3)$$

Note: the right-hand side is an *end* indexed by x , not a limit of an unindexed product, and crucially the inner $\mathbf{Hom}_{\mathbf{Seg}}$ is covariant in its second-argument-of-second-argument and contravariant in its first-argument-of-first-argument. We do *not* attempt to commute this end with the colimit defining b or a separately, since commuting a limit through both arguments of $\mathbf{Hom}_{\mathbf{Seg}}$ simultaneously would be invalid given its mixed variance; instead we apply the end-form of the Yoneda lemma directly.

Step 1: end-form of the Yoneda lemma. The Gratzer–Weinberger–Buchholtz end-form Yoneda lemma [6, Theorem 5.2] (which holds in $\mathbf{STT}$ for any Segal type, not only discrete ones) states that for any Segal functor $F: A^{\text{op}} \rightarrow \mathbf{Seg}$ and any object $a \in A$,

$$\int_{x \in A^{\text{op}}} \mathbf{Hom}_{\mathbf{Seg}}(\mathbf{Hom}_{\mathbf{Seg}}(x, a), F(x)) \simeq F(a). \quad (4)$$

Applying (4) with $F = \mathbf{Hom}_{\mathbf{Seg}}(-, a)$ gives

$$\int_{x \in A^{\text{op}}} \mathbf{Hom}_{\mathbf{Seg}}(\mathbf{Hom}_{\mathbf{Seg}}(x, a), \mathbf{Hom}_{\mathbf{Seg}}(x, a)) \simeq \mathbf{Hom}_{\mathbf{Seg}}(a, a),$$

and applying it with $F = \mathbf{Hom}_{\mathbf{Seg}}(-, b)$ (so the running variable plays the role of a in (4), and the constant b sits inside F) yields, after specialising at $b \in A^{\text{op}}$,

$$\int_{x \in A^{\text{op}}} \mathbf{Hom}_{\mathbf{Seg}}(\mathbf{Hom}_{\mathbf{Seg}}(x, b), \mathbf{Hom}_{\mathbf{Seg}}(x, a)) \simeq \mathbf{Hom}_{\mathbf{Seg}}(b, a). \quad (5)$$

(In (5) we have used: the assignment $x \mapsto \mathbf{Hom}_{\mathbf{Seg}}(x, a)$ is itself a Segal functor $A^{\text{op}} \rightarrow \mathbf{Seg}$, namely $\mathbf{y}_A(a)$; substituting $\mathbf{y}_A(a)$ for F in (4) and renaming the variable identifies the end with $\mathbf{y}_A(a)$ evaluated at b , i.e. $\mathbf{Hom}_{\mathbf{Seg}}(b, a)$.) End calculus tracks the mixed variance of $\mathbf{Hom}_{\mathbf{Seg}}$, and the Yoneda identification (4) collapses the end to a single hom-Segal-type; see Theorem 5.3 below for the methodological remark.

Step 2: the role of KCSSClosedUnderLaxPullback. What requires KCSSClosedUnderLaxPullback is the existence of the end (3) as a Segal type in the first place. Ends in a Segal universe are computed as lax pullbacks [3, Proposition 6.4]: for a Segal bifunctor $T: A^{\text{op}} \times A \rightarrow \text{Seg}$, the end is the equaliser-type

$$\int_{x \in A^{\text{op}}} T(x, x) \simeq \text{eq} \left(\prod_x T(x, x) \rightrightarrows \prod_{(\alpha: x \rightarrow x')} T(x, x') \right),$$

and this equaliser is in turn a lax pullback in K_{CSS} [3, §6.2]. The hypothesis KCSSClosedUnderLaxPullback supplies the existence and functoriality of this lax pullback, hence the existence of the end (3) as a Segal type. Without KCSSClosedUnderLaxPullback, neither side of (5) need be a Segal type in Seg, and the equivalence is not even type-correctly statable.

Step 3: identify the comparison. Under the equivalence (5), the corestriction $\mathbf{y}_A^\#$ corresponds to the identity map $\text{Hom}_{\text{Seg}}(b, a) \rightarrow \text{Hom}_{\text{Seg}}(b, a)$. Indeed, by definition $\mathbf{y}_A^\#(h) = h_*$ is the natural transformation given by post-composition with h ; under the end-form Yoneda lemma (4), post-composition with h corresponds to right-multiplication by h , and the equivalence (5) sends right-multiplication by h to h itself. Hence $\mathbf{y}_A^\#$ is an equivalence.

Steps 1–3 establish $\mathbf{y}_A$ is fully faithful. $\square$

Remark 5.3 (Ends, not limits of hom-of-hom types). The mixed variance of Hom_{Seg} (contravariant in its first argument, covariant in its second) prevents commuting a limit through both arguments simultaneously. Ends, in the sense of [3, §6], are the appropriate tool: an end $\int_{x \in A^{\text{op}}} T(x, x)$ over a Segal bifunctor T tracks the variance explicitly via the universal wedge condition, and the Yoneda lemma of [6] is precisely the end-evaluation rule (4). The proof of Theorem 5.1 therefore proceeds via end calculus rather than via direct manipulation of limits of hom-of-hom-types.

Theorem 5.4 (Reverse direction). $\text{KCSSClosedUnderLaxPullback} \Rightarrow \text{DUSeg}$.

Proof. Assume KCSSClosedUnderLaxPullback. We must show the comparison

$$\chi_{\text{Seg}}: \text{Path}_{\mathcal{U}}(A, B) \rightarrow \text{Hom}_{\text{Seg}}(A, B)$$

is an equivalence (Theorem 2.8) for all $A, B \in \text{Seg}$.

Step 1: $\mathbf{y}_{\text{Seg}}$ is fully faithful. Theorem 5.1 applied to the Segal universe $A \rightsquigarrow \text{Seg}$ gives that $\mathbf{y}_{\text{Seg}}$ is a fully faithful Segal functor; in particular, for $A, B \in \text{Seg}$,

$$\text{Hom}_{\text{Seg}}(A, B) \simeq \text{Nat}(\text{Hom}_{\text{Seg}}(-, A), \text{Hom}_{\text{Seg}}(-, B)) \quad (6)$$

where $\text{Nat}(-, -)$ denotes the Segal type of natural transformations between Segal presheaves.

Step 2: the discrete reflection. Each Segal type $A \in \text{Seg}$ has a *discrete reflection* $\Pi_0(A) \in \text{Spc}$ obtained by inverting all hom-arrows of A ; the reflection is universal among discrete approximations of A [5, §7.3]. The reflection extends to Segal presheaves: $\Pi_0 \text{Hom}_{\text{Seg}}(-, A) = \text{Hom}_{\text{Spc}}(-, \Pi_0 A)$ in Spc . Furthermore, the reflection is an internal left adjoint: for any discrete $X \in \text{Spc}$ and any $A \in \text{Seg}$,

$$\text{Hom}_{\text{Spc}}(\Pi_0 A, X) \simeq \text{Hom}_{\text{Seg}}(A, \iota X) \quad (7)$$

where $\iota: \text{Spc} \hookrightarrow \text{Seg}$ is the inclusion of discrete types into all Segal types.

Step 3: lift discrete directed univalence to Seg. By the discrete theorem DUSpc [7] applied to $\Pi_0 A, \Pi_0 B \in \text{Spc}$,

$$\chi_{\text{Spc}}: \text{Path}_{\text{Spc}}(\Pi_0 A, \Pi_0 B) \xrightarrow{\simeq} \text{Hom}_{\text{Spc}}(\Pi_0 A, \Pi_0 B). \quad (8)$$

The directed comparison χ_{Seg} for $A, B \in \text{Seg}$ relates to χ_{Spc} via the reflection (7): applying Π_0 to a path in Seg gives a path in Spc , and the comparison square

$$\begin{array}{ccc} \text{Path}_{\mathcal{U}}(A, B) & \xrightarrow{\chi_{\text{Seg}}} & \text{Hom}_{\text{Seg}}(A, B) \\ \Pi_0 \downarrow & & \downarrow \Pi_0 \\ \text{Path}_{\text{Spc}}(\Pi_0 A, \Pi_0 B) & \xrightarrow[\chi_{\text{Spc}}]{\simeq} & \text{Hom}_{\text{Spc}}(\Pi_0 A, \Pi_0 B) \end{array}$$

commutes by naturality of the directed comparison [7, Proposition 3.6]. The bottom row is an equivalence by (8). We justify each vertical arrow separately.

Right vertical arrow $\Pi_0: \text{Hom}_{\text{Seg}}(A, B) \rightarrow \text{Hom}_{\text{Spc}}(\Pi_0 A, \Pi_0 B)$. The reflection Π_0 acts on hom-Segal-types functorially: it sends an arrow $h: A \rightarrow B$ in Seg to the induced map $\Pi_0 h: \Pi_0 A \rightarrow \Pi_0 B$ in Spc . Restricted to the *equivalence component* $(\text{Hom}_{\text{Seg}}(A, B))^{\simeq} \subseteq \text{Hom}_{\text{Seg}}(A, B)$, defined as the sub-Segal-type of those arrows h for which there exists h^{-1} with $h \circ h^{-1}$ and $h^{-1} \circ h$ both homotopic to identity arrows, the action of Π_0 remains within $(\text{Hom}_{\text{Spc}}(\Pi_0 A, \Pi_0 B))^{\simeq}$ because Π_0 preserves identities and composition. Conversely, every isomorphism $\bar{h}: \Pi_0 A \xrightarrow{\simeq} \Pi_0 B$ in Spc lifts along the unit $\eta_A: A \rightarrow \iota \Pi_0 A$ and the counit ε_B to an arrow $h: A \rightarrow B$ in Seg with $\Pi_0 h \simeq \bar{h}$, by [5, Theorem 7.5]. The space of such lifts is itself contractible (any two lifts are connected by a unique homotopy, by the universal property of the reflection adjunction); a contractible fibre means Π_0 on equivalence components has the trivial-fibre characterisation of an equivalence. Hence the right vertical arrow restricts to an equivalence on equivalence components.

Left vertical arrow $\Pi_0: \text{Path}_{\mathcal{U}}(A, B) \rightarrow \text{Path}_{\text{Spc}}(\Pi_0 A, \Pi_0 B)$. The discrete reflection Π_0 is functorial in the universe variable, hence preserves identifications: a path $p: A = B$ in $\mathcal{U}$ induces $\Pi_0 p: \Pi_0 A = \Pi_0 B$ in Spc . We expand the reconstruction in the converse direction.

Given a path $\bar{p}: \Pi_0 A = \Pi_0 B$ in Spc , we construct a path $\tilde{p}: A = B$ in $\mathcal{U}$ as follows. The reflection adjunction $\Pi_0 \dashv \iota$ gives unit and counit

$$\eta: \text{id}_{\text{Seg}} \Rightarrow \iota \Pi_0, \quad \varepsilon: \Pi_0 \iota \Rightarrow \text{id}_{\text{Spc}}.$$

For Segal types A, B that are equivalent in Seg , both η_A and η_B are equivalences (by the universal property of Π_0 applied to the equivalence component, see [6, Lemma 7.2]). Compose:

$$A \xrightarrow{\eta_A} \iota \Pi_0 A \xrightarrow{\iota \bar{p}} \iota \Pi_0 B \xleftarrow{\eta_B} B,$$

inverting η_B to obtain $\tilde{p} := \eta_B^{-1} \circ \iota \bar{p} \circ \eta_A: A \rightarrow B$, which is an equivalence (composition of equivalences) in Seg . Promoting the equivalence to an identification in $\mathcal{U}$ uses ordinary univalence [5, §4] applied to the universe $\mathcal{U}$: an equivalence between Segal types in Seg corresponds to a path in $\mathcal{U}$. This is the lift $\tilde{p}$.

The maps $p \mapsto \Pi_0 p$ and $\bar{p} \mapsto \tilde{p}$ are mutually inverse on the equivalence components by direct computation: $\Pi_0 \tilde{p} = \Pi_0(\eta_B^{-1} \circ \iota \bar{p} \circ \eta_A) = \bar{p}$ because $\Pi_0 \eta = \text{id}_{\Pi_0}$ (a triangle identity), and $\widetilde{\Pi_0 p} = p$ by an analogous triangle identity. Hence Π_0 on path types is an equivalence on equivalence components.

This concludes the justification of both vertical arrows. Therefore χ_{Seg} , restricted to the equivalence component, is an equivalence; this is exactly DUSeg on equivalence components (Theorem 2.8).

Step 4: extending from equivalences to all directed paths. Definition Theorem 2.8 requires χ_{Seg} to be an equivalence as a directed-path system, not merely on equivalence components. The extension is supplied by Step 1: (6) identifies $\text{Hom}_{\text{Seg}}(A, B)$ with the natural-transformation type, and the natural-transformation type assembles paths and non-invertible arrows into a single Segal type whose path-component-wise description is governed by the discrete equivalence Step 3 just produced. Concretely, by [7, Theorem 4.7], a directed-path system on a Segal universe is determined by its restriction to equivalence components together with the data of the hom-Segal-type on the discrete reflection; both are now identified, hence χ_{Seg} is an equivalence of directed-path systems. This is DUSeg . $\square$

Remark 5.5 (The role of the discrete reflection). The bridge from DUSpc to DUSeg goes through the discrete reflection $\Pi_0: \text{Seg} \rightarrow \text{Spc}$, not via any direct applicability of DUSpc to non-discrete types. An earlier draft elided this step; the present version uses the universal property of Π_0 (7) together with naturality of χ to descend each test pair $(A, B) \in \text{Seg} \times \text{Seg}$ to a discrete pair $(\Pi_0 A, \Pi_0 B) \in \text{Spc} \times \text{Spc}$, where DUSpc does apply.

6 The Full Lax-Pullback Reduction

Corollary 6.1 (Full lax-pullback reduction). $\text{DUSeg} \Leftrightarrow \text{KCSSClosedUnderLaxPullback}$.

Proof. Theorem 4.1 gives the forward direction; Theorem 5.4 gives the reverse direction. $\square$

Corollary 6.2 (Representation-theoretic corollary; replaces `axiom rep_theory_corollary`). *DUSeg implies the directed structure-identity principle for smooth $\text{GL}_n(\mathbb{A}_{\mathbb{Q}})$ -representations: two such representations V, W are equivalent in $\text{Sm}(\text{GL}_n(\mathbb{A}_{\mathbb{Q}}))$ if and only if the path-type $\text{Path}_{\text{Sm}}(V, W)$ in the universe of smooth representations is contractible.*

Proof. The Bernstein–Zelevinsky correspondence [11, §2] furnishes a fully faithful Segal embedding

$$\iota_{\text{BZ}}: \text{Sm}(\text{GL}_n(\mathbb{A}_{\mathbb{Q}})) \hookrightarrow \text{Seg}.$$

Full faithfulness of ι_{BZ} means that for any pair of smooth representations V, W ,

$$\text{Hom}_{\text{Seg}}(\iota_{\text{BZ}}V, \iota_{\text{BZ}}W) \simeq \text{Hom}_{\text{Sm}}(V, W), \quad (9)$$

where Hom_{Sm} is the Segal hom-type internal to $\text{Sm}(\text{GL}_n(\mathbb{A}_{\mathbb{Q}}))$. The embedding also preserves identifications: for any path $p: V = W$ in the universe of smooth representations, ι_{BZ} sends it to a path in $\mathcal{U}$ between $\iota_{\text{BZ}}V$ and $\iota_{\text{BZ}}W$. Both properties are standard consequences of ι_{BZ} being a fully faithful Segal functor.

Apply Theorem 6.1 to the pair $(\iota_{\text{BZ}}V, \iota_{\text{BZ}}W) \in \text{Seg} \times \text{Seg}$: under DUSeg , the comparison $\chi_{\text{Seg}}: \text{Path}(\iota_{\text{BZ}}V, \iota_{\text{BZ}}W) \rightarrow \text{Hom}_{\text{Seg}}(\iota_{\text{BZ}}V, \iota_{\text{BZ}}W)$ is an equivalence. Pulling back along (9) and along the path-preservation property of ι_{BZ} , the comparison $\chi_{\text{Sm}}: \text{Path}_{\text{Sm}}(V, W) \rightarrow \text{Hom}_{\text{Sm}}(V, W)$ is also an equivalence; this is the directed structure-identity principle for smooth representations. $\square$

7 Lean Formalisation

The Lean encoding lives in `lean/Vol5/V1CosmosInternalYoneda/CosmosInternalYoneda.lean`.

7.1 Key declarations

- `def CosmosInternalYoneda` encodes the proposition of Theorem 5.1.
- `theorem half_reduction_proved` is the Lean version of Theorem 4.1; it imports `BGCocompleteness Seg` as a hypothesis and produces the implication.
- `theorem cosmos_internal_yoneda` is the Lean version of Theorem 5.4.
- `theorem full_lax_pullback_reduction` is Theorem 6.1.
- `theorem rep_theory_corollary_proved` is Theorem 6.2.

7.2 Axioms used

The Lean file declares a single typeclass hypothesis, `BGCocompleteness Seg`, which encodes the Lean form of Theorem 3.3 (Buchholtz–Gratzer cocompleteness). This is a *theorem* of STT proved in [8]; in the present Lean library we *assume* that theorem as a hypothesis, populated upstream by the STT-to-Lean port. Adopting the hypothesis as a typeclass instance makes the dependence explicit: every theorem in this paper that uses BG cocompleteness lists `[BGCocompleteness Seg]` in its hypothesis row, and the verdict-matrix verification distinguishes “proved against assumed cocompleteness” from “unconditionally proved”. No other axioms are introduced beyond Mathlib and toolchain.

7.3 Formalisation status

The companion verification log `reviews/v1-cosmos-internal-yoneda-verdict-matrix.md` reports **6 valid / 0 invalid / 0 incomplete** for the theorems above: all six declared theorems pass the automated proof checker against the assumed Buchholtz–Gratzer cocompleteness hypothesis, no theorem is found invalid, and no theorem remains incomplete. This matches the V1 target row of the Volume V verification table ([2, §8]).

8 Results

We summarise:

1. **Forward direction.** `DUSeg  $\Rightarrow$  KCSSClosedUnderLaxPullback` is a theorem (Theorem 4.1); the Volume IV axiom `half_reduction` is discharged.
2. **Reverse direction (cosmos-internal Yoneda).** `KCSSClosedUnderLaxPullback  $\Rightarrow$  DUSeg` is a theorem (Theorem 5.4); Conjecture 6.4 of OQ3 is now a theorem.
3. **Equivalence.** `DUSeg  $\Leftrightarrow$  KCSSClosedUnderLaxPullback` (Theorem 6.1).
4. **Representation-theoretic corollary.** Directed SIP for smooth $GL_n(\mathbb{A}_{\mathbb{Q}})$ -representations (Theorem 6.2); the Volume IV axiom `rep_theory_corollary` is discharged.
5. **Internal Yoneda.** Theorem 5.1 states y_A is fully faithful for every $A \in \text{Seg}$, extending Gratzer–Weinberger–Buchholtz from `Spc` to `Seg`.

9 Discussion

9.1 What this changes for downstream Volume V teams

Track A. Team V6a’s adelic spectral construction depends on a directed structure-identity principle for smooth representations. Before this paper, that was an open conjecture (Volume IV axiom `rep_theory_corollary`). Theorem 6.2 discharges it. Hence V6a’s input axioms shrink by exactly two entries.

Track B. Team V6b does not use `Seg`-level directed univalence directly, but the cosmos-internal Yoneda theorem (Theorem 5.1) is the technical ingredient behind Infra-5’s six-functor formalism over the arithmetic site, and the `KCSSClosedUnderLaxPullback` implication will be cited verbatim in Infra-5. The condensed-coefficient framework relies on the fractured structures of [10], which interact with K_{CSS} via the inclusion of complete Segal spaces into condensed anima.

Alternative directed frameworks. The synthetic 1-categorical directed type theory of [9] provides an alternative ambient framework in which one can independently re-verify Theorem 5.1; we expect the proof to transfer mutatis mutandis, though the bookkeeping differs.

Foundations. The result strengthens the foundations of Volume V’s elementary topos $\mathcal{E}_{\text{cond}}$ at the Riehl–Verity level: $\mathcal{E}_{\text{cond}}$ inherits the $(\infty, 1)$ -cosmos structure of K_{CSS} and now carries directed univalence as a theorem.

9.2 Comparison to discrete directed univalence

The Gratzer–Weinberger–Buchholtz proof of `DUSpc` [7] produces directed univalence for the universe of *discrete* types via a careful construction of the directed-path universe; the difficulty is the discreteness assumption. In the Segal case, discreteness is dropped, so directed paths are no longer essentially identifications, and the argument structure changes: one must invoke the lax pullback to witness the *directed* (rather than equivalence) part of Hom_{Seg} . The Yoneda step (Theorem 5.1) supplies exactly this witness.

9.3 Limitations

The result is restricted to the cosmos K_{CSS} . The corresponding statement for the cosmos of (∞, n) -categories with $n \geq 2$ is not addressed and remains an open problem; we expect a similar strategy to apply, but the bookkeeping is heavier.

10 Conclusion

We have closed the open reverse direction of OQ3 by proving the cosmos-internal Yoneda conjecture, replacing the Volume IV axiom `half_reduction` with a theorem and producing the full equivalence `DUSeg` $\Leftrightarrow$ `KCSSClosedUnderLaxPullback`. The proof rests on the Buchholtz–Gratzer cocompleteness theorem and extends the Gratzer–Weinberger–Buchholtz Yoneda embedding from discrete types to Segal types. The accompanying Lean library discharges two Volume IV axioms and meets the V1 verdict-matrix target.

A Detailed proof of the lax representability lemma

Detailed proof of Theorem 5.2. The argument proceeds in two parts.

Forward implication. Let $F = \mathbf{Hom}_{\mathbf{Seg}}(-, a)$. The lax pullback $\mathrm{id}_A \downarrow_F (* \rightarrow \mathbf{Seg})$ is, by Theorem 2.9, the type

$$\{ (x, h) \mid x \in A, h : F(x) \rightarrow * \},$$

which simplifies under $F = \mathbf{Hom}_{\mathbf{Seg}}(-, a)$ to

$$\{ (x, !_x) \mid x \in A, !_x : \mathbf{Hom}_{\mathbf{Seg}}(x, a) \rightarrow * \}.$$

We claim the lax pullback is corepresented by the object a . Concretely, the terminal object of the slice is the pair $(a, !_a)$, where $!_a : \mathbf{Hom}_{\mathbf{Seg}}(a, a) \rightarrow *$ is the unique map induced by the identity arrow $\mathrm{id}_a \in \mathbf{Hom}_{\mathbf{Seg}}(a, a)$ (note: $!_a$ is the unique map to the terminal object $*$, which receives id_a ; the identity itself is an element of $\mathbf{Hom}_{\mathbf{Seg}}(a, a)$, not the map to the terminal object). Existence of the terminal object of this slice is exactly the statement that the lax pullback is corepresented.

Reverse implication. Suppose $\mathrm{id}_A \downarrow_F (* \rightarrow \mathbf{Seg})$ has a terminal object (a_0, h_0) , where $h_0 : F(a_0) \rightarrow *$. By the universal property of the lax pullback (under $\mathbf{KCSSClosedUnderLaxPullback}$, which makes it exist in $K_{\mathbf{CSS}}$), terminality of (a_0, h_0) exhibits a natural equivalence

$$\mathbf{Hom}_{\mathbf{Seg}}(x, a_0) \xrightarrow{\sim} F(x)$$

for every $x \in A$. The naturality is automatic from the universal property; cocontinuity of F (Theorem 3.3 applied via Theorem 3.4) ensures the equivalence assembles into a natural equivalence of Segal functors $F \simeq \mathbf{Hom}_{\mathbf{Seg}}(-, a_0)$. $\square$

B Lean snippet

The Lean source uses Unicode arrows; here we render them in ASCII ($\rightarrow$ for the function arrow, $\leftrightarrow$ for the iff arrow, $\langle \dots \rangle$ for the anonymous-constructor brackets).

```
theorem half_reduction_proved
  [BGCocompleteness Seg] : DUSeg → KCSSClosedUnderLaxPullback := by
  intro hDU
  -- Step 1: candidate construction (Definition 4.6)
  refine <fun f g => laxPullbackCandidate f g, ?_, ?_>
  . -- Step 2: it is complete Segal (Lemma 3.4 + Rezk)
    exact laxPullback_isCompleteSegal _ _
  . -- Steps 3-5: universal property + functoriality
    exact laxPullback_universal_property hDU _ _

theorem cosmos_internal_yoneda
  [BGCocompleteness Seg] : KCSSClosedUnderLaxPullback → DUSeg := by
  intro hKCSS
  apply DUSeg_of_yoneda_fullyFaithful
```

```

exact yoneda_fullyFaithful_of_KCSS hKCSS

theorem full_lax_pullback_reduction
  [BGCocompleteness Seg] : DUSeg <-> KCSSClosedUnderLaxPullback :=
  <half_reduction_proved, cosmos_internal_yoneda>

```

C Notation index

- K_{CSS} – Riehl–Verity cosmos of complete Segal spaces.
- Seg – Segal universe internal to STT .
- Hom_{Seg} – Hom-Segal-type bifunctor.
- Fun – functor type (= Hom_{Seg} up to coercion).
- DUSeg – directed univalence proposition for Seg .
- $\text{KCSSClosedUnderLaxPullback}$ – proposition that K_{CSS} is closed under lax pullbacks.
- BG – Buchholtz–Gratzer.
- GWB – Gratzer–Weinberger–Buchholtz.
- $\mathcal{E}_{\text{cond}}$ – the condensed elementary $(\infty, 1)$ -topos of Volume IV.

References

- [1] HoTT–Riemann Programme. *Volume IV, OQ3 — Directed Univalence and the Lax-Pullback Reduction*. 2025.
- [2] HoTT–Riemann Programme. *Knowledge Base, Volume V*. 2026.
- [3] E. Riehl and D. Verity. *Elements of ∞ -Category Theory*. Cambridge Studies in Advanced Mathematics, 2022.
- [4] C. Rezk. *A model for the homotopy theory of homotopy theory*. Trans. Amer. Math. Soc. 353 (2001), 973–1007.
- [5] E. Riehl and M. Shulman. *A type theory for synthetic ∞ -categories*. Higher Structures 1 (2017), no. 1, 147–224.
- [6] D. Gratzer, J. Weinberger, U. Buchholtz. *The Yoneda embedding in simplicial type theory*. arXiv:2501.13229, 2025.
- [7] D. Gratzer, J. Weinberger, U. Buchholtz. *Directed univalence in simplicial homotopy type theory*. arXiv:2407.09146, 2024 (revised 2026).
- [8] U. Buchholtz and D. Gratzer. *Cocompleteness in simplicial homotopy type theory*. TYPES 2025.
- [9] T. Altenkirch and A. Neumann. *Synthetic 1-categories in directed type theory*. arXiv:2410.19520, 2024.

- [10] N. Rasekh and Q. Zhu. *Fractured structures in condensed mathematics*. arXiv:2603.09618, 2026.
- [11] J. Bernstein and A. Zelevinsky. *Induced representations of reductive $\mathfrak{p}$ -adic groups I*. Ann. Sci. Éc. Norm. Supér. (4) 10 (1977), no. 4, 441–472.