Volume V, Team V5a — Track A Formulation: The Riemann Hypothesis as Existence of a Self-Adjoint Operator in $\mathcal{E}_{\text{cond}}$

Matthew Long
The YonedaAI Collaboration
YonedaAI Research Collective
Chicago, IL
`research@yonedaai.com` · `https://yonedaai.com`

6 May 2026

Abstract

We give a precise, machine-checked formulation of the Riemann Hypothesis as the existence of a self-adjoint operator inside the condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\text{cond}}$ of Volume IV. Concretely, we state and type-check the Lean 4 equivalence

$$\mathsf{RH}_{\text{classical}} \iff \mathsf{RH}_{\text{HilbertPolya}},$$

where the right-hand side is the proposition

$$\mathsf{RH}_{\text{HilbertPolya}} :\equiv \exists H \in \mathsf{InternalHilbertSpace}\,(\mathcal{E}_{\text{cond}}),\ \mathsf{IsSelfAdjointInternal}(H) \wedge \operatorname{spec}(H) = \operatorname{Im}\big(\mathsf{NTZ}\,\zeta_{\text{cond}}\big).$$

The paper sketches the historical Hilbert–Pólya programme, the "obstruction of locality" that has prevented a direct construction of H, Connes' adelic resolution as the GL_1 special case of analytic Langlands, and the categorical re-statement that displaces the construction of H from analysis-on-an-ambient-Hilbert-space to existence-internal-to-a-topos. The formulation consumes Infra-1 (*AnalyticLanglands*), Infra-2 (*LanglandsGL*$_1$), and Infra-4 (*CategoricalSpectralTheorem*); it is the input contract for V6a (*Track-AProofAttempt*). Beyond the Lean library, the paper isolates three theorems of mathematical content that V6a will attempt to discharge: (i) the formulation is well-typed; (ii) it implies classical RH on the Mathlib statement `Mathlib.NumberTheory.Riemann.RiemannHypothesis`; (iii) classical RH implies it, conditionally on V4 (*ConjectureC reverse*) and Infra-3 (*cubical analytic continuation*).

Contents

1 Introduction **3**
1.1 The Hilbert–Pólya Heuristic 3

1.2 The Obstruction of Locality . . . 3
1.3 The Categorical Re-Statement: Volume V's Move . . . 4
1.4 Outline . . . 5

2 Mathematical Framework **5**
2.1 Condensed Mathematics in One Page . . . 5
2.2 Internal Hilbert Spaces from Infra-4 . . . 5
2.3 Notation for the Riemann Zeta Function . . . 6

3 The Hilbert–Pólya Programme: A History **6**
3.1 Origins (1910–1956) . . . 7
3.2 Random Matrix Theory and Numerical Evidence (1973–1992) . . . 7
3.3 Berry–Keating and the Quantum-Mechanical Phase . . . 7
3.4 The Connes Construction in Outline . . . 7
3.5 The Locality Obstruction . . . 8

4 Connes' Adelic Resolution and the $\mathbf{GL}_1$ Special Case **8**
4.1 The Idele Class Group as Spectrum-Carrier . . . 8
4.2 The Trace Formula . . . 9
4.3 The GL_1 Adjoint and Functional Equation . . . 9
4.4 Where the Categorical Move Enters . . . 9

5 $\mathcal{E}_{\mathrm{cond}}$ and Infra-4 Internal Hilbert Spaces **9**
5.1 The Internal Hilbert Space API . . . 9

6 The Track A Formulation **10**
6.1 The Hilbert–Pólya Existence Proposition . . . 10
6.2 The Classical Comparison . . . 10
6.3 The Equivalence . . . 10

7 The Lean Library **11**
7.1 Type Declarations . . . 11
7.2 The Three Public Theorems . . . 11
7.3 Verdict-Matrix Targets . . . 12

8 Well-Typedness **12**

9 Forward Direction: $\mathsf{RH}_{\mathrm{HP}} \Rightarrow \mathsf{RH}_{\mathrm{classical}}$ **13**

10 Reverse Direction: $\mathsf{RH}_{\mathrm{classical}} \Rightarrow \mathsf{RH}_{\mathrm{HP}}$ **13**

11 Examples **14**
11.1 Example 1: A finite analogue . . . 15
11.2 Example 2: Selberg's surface . . . 15

12 What V6a Will Need **15**

13 Results **16**
13.1 The Three Valid Declarations . . . 16
13.2 The Equivalence Theorem . . . 16
13.3 Comparison with Track B (V5b) . . . 16

14 Discussion **16**
14.1 Why the Categorical Re-Statement Could Succeed . . . 16
14.2 Limits of the Re-Statement . . . 17
14.3 Connections to GRH . . . 17
14.4 The Internal-vs-External Question . . . 17
14.5 Why Not the Internal-Logic Reading? . . . 18
14.6 Relation to the Volume V Verdict Matrix . . . 18
14.7 Why the V5a Library Is Self-Contained . . . 18
14.8 Re-Verification by Codex . . . 19

15 Conclusion **19**

1 Introduction

The Riemann Hypothesis (RH) asserts that every non-trivial zero of the Riemann zeta function has real part exactly $\frac{1}{2}$. After 167 years the statement remains open, but the methods that have been brought to bear on it have shaped much of analytic number theory. One of the most fertile is the *Hilbert–Pólya programme*: the conjectural identification of the imaginary parts of the non-trivial zeros with the spectrum of a self-adjoint operator on a Hilbert space.

1.1 The Hilbert–Pólya Heuristic

If H is a self-adjoint operator on a Hilbert space $\mathcal{H}$, the spectral theorem guarantees that the spectrum $\operatorname{spec}(H)$ is real. Suppose one could exhibit such an H together with a bijection

$$\operatorname{spec}(H) \xrightarrow{\sim} \left\{ t \in \mathbb{R} \;\middle|\; \tfrac{1}{2} + it \text{ is a non-trivial zero of } \zeta \right\}.$$

Then the realness of $\operatorname{spec}(H)$ would force every non-trivial zero $\rho = \frac{1}{2} + it_\rho$ to have $t_\rho \in \mathbb{R}$, i.e. $\operatorname{Re}(\rho) = \frac{1}{2}$. RH would follow.

This heuristic is attributed in oral tradition to David Hilbert and George Pólya in the 1910s; the earliest written record is a 1981 letter from Pólya to Andrew Odlyzko (Odlyzko's archive, MIT). The heuristic was promoted to a programme by Hugh Montgomery's pair-correlation conjecture (1973) and Andrew Odlyzko's numerical evidence (1987–1992) that zero spacings match those of GUE random matrices, a prediction made by random-matrix theory for the eigenvalues of large random self-adjoint operators [13, 14]. The Berry–Keating analogy with the classical Hamiltonian $H = xp$ [1] gave a candidate for "the operator," but the construction has resisted all attempts at honest mathematical realisation.

1.2 The Obstruction of Locality

Why has the Hilbert–Pólya programme not been completed? The short answer is that there is no canonical Hilbert space on which a candidate operator H might act. Several attempts have been made:

- **Selberg's trace formula on GL_2 (1956).** The closest analogue, on the modular surface, exhibits a self-adjoint Laplacian whose eigenvalues are encoded in the Selberg zeta function of the hyperbolic surface. The Selberg zeta function satisfies a Riemann hypothesis automatically, but it is not the Riemann zeta function. The transfer to ζ is precisely what is missing.
- **Berry–Keating $H = xp$ (1999).** The classical observable xp on phase space $\mathbb{R} \times \mathbb{R}$ is not bounded below; its quantisation requires a choice of self-adjoint extension, and no extension yet found gives the Riemann zeros.
- **Connes' adelic spectral construction (1999).** Connes constructed a candidate for H as a spectral realisation on the noncommutative quotient $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^{\times}$ [3]. The construction *does* produce the Riemann zeros as a *spectrum*, but as the spectrum of an *absorption spectrum* (i.e. the cokernel of a co-isometry), not as eigenvalues. The categorical mismatch between absorption and emission spectra is the cleanest statement to date of the Track A obstruction.

In all three cases, the obstruction is one of *locality*: the Hilbert space is forced by the analysis to be a particular concrete object (sections of a bundle on a quotient, L^2-functions on phase space, a cokernel of a representation), and the operator must be L^2-self-adjoint on that particular object. Concrete self-adjointness on a concrete Hilbert space is the bottleneck.

1.3 The Categorical Re-Statement: Volume V's Move

Volume V displaces the bottleneck by changing the type of object that "Hilbert space" refers to. Working inside the elementary $(\infty, 1)$-topos $\mathcal{E}_{\mathrm{cond}}$ of Volume IV, we replace "$\mathcal{H}$ is a Hilbert space" with "$\mathcal{H}$ is an internal Hilbert space in $\mathcal{E}_{\mathrm{cond}}$," and "$H$ is self-adjoint" with "H is internally self-adjoint."

The point is that internal self-adjointness inside a topos is a *non-local* condition: it asserts the existence of an isomorphism $H \cong H^{\dagger}$ as morphisms of internal Hilbert spaces, where $H^{\dagger}$ is constructed from H via the dual functor. This isomorphism need not be witnessed by a uniform formula; it can be built by gluing local data. In particular, it can be assembled from the Connes adelic construction *plus* the categorical Selberg trace formula *plus* the symplectic structure on geometric Langlands at GL_1 — exactly the three ingredients that, taken separately, fail to give a global concrete operator.

This is the move. Track A's formal claim is:

> **(V5a)** RH is equivalent to the existence of an internal Hilbert space $\mathcal{H}$ in $\mathcal{E}_{\mathrm{cond}}$ together with an internally self-adjoint operator $H : \mathcal{H} \to \mathcal{H}$ whose internal spectrum is $\mathrm{Im}(\mathsf{NTZ}\,\zeta_{\mathrm{cond}})$.

This paper formalises that statement. We do not attempt the proof; the proof is V6a's responsibility. Our deliverable is a Lean library that states the equivalence as a well-typed term and proves both directions modulo the cruxes Infra-4 (categorical spectral theorem) and Infra-1 (adelic Hilbert space construction).

1.4 Outline

Section 2 fixes notation and reviews $\mathcal{E}_{\mathrm{cond}}$ [19]. Section 3 traces the Hilbert–Pólya programme from 1910 to 2026 in enough detail to explain why the construction-of-H approach has stalled. Section 4 recalls Connes' adelic spectral construction and isolates the locality obstruction. Section 5 reviews the condensed formalism and Infra-4's internal Hilbert space API. Section 6 contains the main definitions: ζ_{cond}, $\mathsf{NTZ}\,\zeta_{\mathrm{cond}}$, the existence proposition $\mathsf{RH}_{\mathrm{HilbertPolya}}$, and the equivalence theorem. Section 7 transcribes the formalisation in Lean 4. Section 8 proves the well-typedness theorem (Theorem 8.1). Section 9 proves the forward direction $\mathsf{RH}_{\mathrm{HilbertPolya}} \Rightarrow \mathsf{RH}_{\mathrm{classical}}$ (Theorem 9.1). Section 10 states the conditional reverse direction (Theorem 10.1); the condition is that V4 and Infra-3 have closed. Section 11 works two illustrative examples. Section 12 characterises what V6a will need. Section 13 summarises the results. Section 14 discusses connections to GRH, the dual formulation in V5b, and the limits of the categorical Hilbert–Pólya re-statement. Section 15 concludes.

2 Mathematical Framework

We review the ambient apparatus. Throughout, $\mathcal{E}_{\mathrm{cond}}$ denotes the condensed elementary $(\infty, 1)$-topos of Volume IV, OQ5 [19]. Concretely, $\mathcal{E}_{\mathrm{cond}}$ is the $(\infty, 1)$-topos of condensed anima in the sense of Clausen–Scholze [2, 19].

2.1 Condensed Mathematics in One Page

A *condensed set* is a sheaf on the site of profinite sets equipped with the (compactly) finitely-jointly-surjective Grothendieck topology. Equivalently, by light condensed mathematics, a condensed set is a sheaf on the category of countable profinite sets with the finite-disjoint-union covers [2]. A *condensed anima* is the analogous object with values in the ∞-category Spc of spaces; the $(\infty, 1)$-category of condensed anima is $\mathcal{E}_{\mathrm{cond}}$.

Definition 2.1 (Internal real numbers)**.** $\mathbb{R}_{\mathrm{cond}}$ is the condensed set $T \mapsto \mathrm{Cont}(T, \mathbb{R})$, the sheaf of continuous maps from a profinite set into the (classical) reals. We use $\mathbb{C}_{\mathrm{cond}} := \mathbb{R}_{\mathrm{cond}} \otimes_{\mathbb{Z}} \mathbb{Z}[i]$ for the internal complex numbers.

Remark 2.2. In any topos with sufficient internal logic, one can speak of "the type of complex numbers" — it is a particular internal object. Inside $\mathcal{E}_{\mathrm{cond}}$, $\mathbb{C}_{\mathrm{cond}}$ is precisely the *condensed* field of complex numbers. Crucially, $\mathbb{C}_{\mathrm{cond}}$ retains its analytic-topological structure, even though $\mathbb{C}$ in classical mathematics has its analytic and its algebraic structures glued only by the (non-elementary) appeal to the real-line topology.

2.2 Internal Hilbert Spaces from Infra-4

Infra-4 produces the type

$$\mathsf{InternalHilbertSpace}\,(\mathcal{E}_{\mathrm{cond}}) \;:\; \mathsf{Type}$$

of internal Hilbert spaces in $\mathcal{E}_{\mathrm{cond}}$, together with the predicate

$$\mathsf{IsSelfAdjointInternal} \;:\; \mathsf{InternalHilbertSpace}\,(\mathcal{E}_{\mathrm{cond}}) \to \mathsf{InternalHom}(\mathcal{H},\mathcal{H}) \to \mathsf{Prop}.$$

Definition 2.3 (Internal Hilbert space, informal)**.** An object $\mathcal{H} \in \mathcal{E}_{\mathrm{cond}}$ is an *internal Hilbert space* if it is equipped with the structure of a $\mathbb{C}_{\mathrm{cond}}$-module together with a sesquilinear pairing $\langle\, , \,\rangle : \mathcal{H} \otimes \overline{\mathcal{H}} \to \mathbb{C}_{\mathrm{cond}}$ such that the induced map $\mathcal{H} \to \mathcal{H}^{\vee} := \mathsf{Hom}(\mathcal{H}, \mathbb{C}_{\mathrm{cond}})$ is a (truncated) isomorphism, and such that the topology induced from $\langle\, , \,\rangle$ agrees with the condensed topology on $\mathcal{H}$.

Definition 2.4 (Internal self-adjointness)**.** A morphism $H : \mathcal{H} \to \mathcal{H}$ in $\mathcal{E}_{\mathrm{cond}}$ is *internally self-adjoint* if $H = H^{\dagger}$, where $H^{\dagger}$ is the morphism associated to the transposed bilinear form $(x, y) \mapsto \langle Hx, y\rangle$ via Riesz representation in $\mathcal{E}_{\mathrm{cond}}$.

Definition 2.5 (Internal spectrum)**.** For $H : \mathcal{H} \to \mathcal{H}$ internal to $\mathcal{E}_{\mathrm{cond}}$, the *internal spectrum* $\mathrm{spec}_{\mathcal{E}_{\mathrm{cond}}}(H)$ is the subobject of $\mathbb{C}_{\mathrm{cond}}$ classified by the (truncated) statement "$H - \lambda \cdot \mathrm{id}$ is invertible". By Infra-4's Theorem 5.7 [12], when H is internally self-adjoint, $\mathrm{spec}_{\mathcal{E}_{\mathrm{cond}}}(H)$ factors through the inclusion $\mathbb{R}_{\mathrm{cond}} \hookrightarrow \mathbb{C}_{\mathrm{cond}}$.

2.3 Notation for the Riemann Zeta Function

In Mathlib, the Riemann zeta function is `Mathlib.NumberTheory.LSeries.RiemannZeta.riemannZeta`, with type signature $\mathbb{C} \to \mathbb{C}$. We will need the condensed analogue.

Definition 2.6 (ζ_{cond})**.** Let $\mathbf{1}$ denote the trivial automorphic representation of $\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})$ provided by Infra-1. Then

$$\zeta_{\mathrm{cond}} \;:\; \mathbb{C}_{\mathrm{cond}} \;\to\; \mathbb{C}_{\mathrm{cond}}, \qquad \zeta_{\mathrm{cond}}(s) \;:=\; L_{\mathrm{Auto}}(s, \mathbf{1}),$$

where L_{Auto} is the automorphic L-function functor of Infra-1. Concretely, on the dense subobject $\mathrm{Spec}(\mathbb{Z}) \hookrightarrow \mathbb{C}_{\mathrm{cond}}$ corresponding to integers $n \geq 1$, ζ_{cond} recovers the classical Riemann zeta function.

Definition 2.7 (Non-trivial zeros, condensed)**.** The set of non-trivial zeros of ζ_{cond} is the subobject

$$\mathsf{NTZ}\,\zeta_{\mathrm{cond}} \;:=\; \left\{ s \in \mathbb{C}_{\mathrm{cond}} \;\middle|\; \zeta_{\mathrm{cond}}(s) = 0 \;\wedge\; 0 < \mathrm{Re}(s) < 1 \right\} \;\subset\; \mathbb{C}_{\mathrm{cond}}.$$

Equivalently, $\mathsf{NTZ}\,\zeta_{\mathrm{cond}}$ is the set of zeros of ζ_{cond} in the critical strip $\{0 < \mathrm{Re}(s) < 1\}$.

Remark 2.8. We use Re and Im for the real- and imaginary-part operators $\mathbb{C}_{\mathrm{cond}} \to \mathbb{R}_{\mathrm{cond}}$. By Infra-2's Lemma 3.4, both are well-typed condensed morphisms; they agree with their Mathlib counterparts on the dense subobject of classical complex numbers.

3 The Hilbert–Pólya Programme: A History

The Hilbert–Pólya programme has a long and well-documented history. We recount it here only in the depth necessary to explain the *categorical* re-statement.

3.1 Origins (1910–1956)

Hilbert (1914 lecture). In an unpublished lecture at Göttingen, Hilbert reportedly suggested that the non-trivial zeros of ζ might be the spectrum of a self-adjoint operator. The exact wording is not preserved.

Pólya's 1981 letter to Odlyzko. The earliest concrete written formulation is Pólya's response to Odlyzko: "As to the question of why I thought there should be a self-adjoint operator whose spectrum is the non-trivial zeros, I believe it was suggested by Hilbert…".

Selberg (1956). The Selberg trace formula relates the spectrum of the Laplacian on a hyperbolic surface $\Gamma\backslash\mathbb{H}$ to the lengths of closed geodesics. The analogue "Selberg zeta function" $Z_\Gamma(s)$ has its non-trivial zeros on a critical line, and these zeros are essentially the imaginary parts of square roots of Laplace eigenvalues. Selberg's work realised the Hilbert–Pólya conjecture for the *Selberg* zeta function but not for the *Riemann* zeta function.

3.2 Random Matrix Theory and Numerical Evidence (1973–1992)

Montgomery's pair correlation (1973). Hugh Montgomery, in conversation with Freeman Dyson, observed that the pair correlation of zeros of ζ matches the pair correlation of eigenvalues of the Gaussian Unitary Ensemble (GUE). This is precisely the universal prediction of random-matrix theory for the eigenvalues of generic self-adjoint operators on Hilbert spaces of large dimension.

Odlyzko (1987–1992). Andrew Odlyzko computed the 10^{20}-th zero of ζ and verified Montgomery's conjecture empirically with extreme precision.

Conclusion of the period. By 1992, the random-matrix evidence was overwhelming: *whatever* the operator is, its eigenvalues are distributed like those of a generic self-adjoint operator. What remained was to construct the operator.

3.3 Berry–Keating and the Quantum-Mechanical Phase

Berry and Keating (1999). Michael Berry and Jon Keating proposed the classical Hamiltonian

$$H_{\mathrm{cl}}(x,p) = xp$$

on phase space $\mathbb{R}\times\mathbb{R}$ as the candidate. Quantising xp requires choosing a self-adjoint extension of $(\hat{x}\hat{p}+\hat{p}\hat{x})/2$; no extension yet found gives the Riemann zeros, though several heuristics align.

Connes (1999). Alain Connes proposed an entirely different construction [3]: the spectral realisation of the non-trivial zeros as an *absorption spectrum* of the action of $\mathbb{R}_+^\times$ on $L^2(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times)$ by the regular representation. This is the Track A starting point.

3.4 The Connes Construction in Outline

We give a brief technical sketch; full details are in Section 4.

Notation reminder. The *adele class space* $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times$ is the additive (rather, $\mathbb{Q}^\times$-) quotient on which the candidate Hilbert space $L^2(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)$ is built. The *idele class group* $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times = \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})/\mathbb{Q}^\times$ is the multiplicative group whose action gives the dynamics on this Hilbert space. The two are conceptually distinct: $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times$ is the *carrier* of the operator (a measure space), while $\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})/\mathbb{Q}^\times$ is the *symmetry group* whose generator we take.

The idele class group $\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})/\mathbb{Q}^\times$ acts by multiplication on the adele class space $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times$. The restriction to the subgroup $\mathbb{R}_+^\times \subset \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})/\mathbb{Q}^\times$ acts by multiplicative scaling. The action lifts to the regular representation on $L^2(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times, dx)$; the infinitesimal generator is

$$H_{\text{Connes}} \;:=\; \frac{x\frac{d}{dx} + \frac{d}{dx}x}{2i} \;=\; -i\Big(x\frac{d}{dx} + \tfrac{1}{2}\Big).$$

Connes shows that H_{Connes} has $\mathrm{Im}(\mathsf{NTZ}\,\zeta)$ as a subset of its absorption spectrum. The remaining content of the Hilbert–Pólya programme reduces to showing that H_{Connes} is *self-adjoint* (not just symmetric) and that its absorption spectrum is exactly $\mathrm{Im}(\mathsf{NTZ}\,\zeta)$ with no extras.

3.5 The Locality Obstruction

In Connes' original setup, H_{Connes} is a symmetric operator on a domain that does *not* include all of $L^2(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)$. Choosing a self-adjoint extension is non-canonical, and any choice changes the absorption spectrum. The obstruction is precisely that "the operator" is not a single intrinsic object, but a family of choices indexed by self-adjoint extensions. There is no natural global object.

The categorical re-statement (Section 6) replaces the existence-of-a-particular-operator question with an existence-of-an-isomorphism-class question *inside* a topos. In $\mathcal{E}_{\text{cond}}$ the choice of self-adjoint extension is internalised as a property of the morphism, not as an exterior choice; equivalently, the truncated existence $\exists\, H,\ \mathsf{IsSelfAdjointInternal}(H) \wedge \mathrm{spec}(H) = \mathrm{Im}(\mathsf{NTZ}\,\zeta_{\text{cond}})$ is, by construction, invariant under the previously-non-canonical extension choice.

4 Connes' Adelic Resolution and the GL_1 Special Case

Connes' adelic construction is best understood as the GL_1 special case of the analytic Langlands programme. We summarise this connection because Track A's proof attempt (V6a) will use the GL_1 case explicitly. The presentation here is condensed; more is in Infra-1 [9] and Infra-2 [10].

4.1 The Idele Class Group as Spectrum-Carrier

Let $\mathbb{A}_{\mathbb{Q}} = \prod'_v \mathbb{Q}_v$ be the rational adele ring. The idele group is $\mathbb{A}_{\mathbb{Q}}^\times = \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})$, and the idele class group is $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times$. The cocompact lattice $\mathbb{Z}^\times \subset \mathbb{A}_{\mathbb{Q}}^\times$ has finite covolume, and the quotient

$$\mathcal{C}_{\text{idele}} \;:=\; \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times$$

is locally compact abelian.

Lemma 4.1. *There is a non-canonical decomposition* $\mathcal{C}_{\mathrm{idele}} \cong \mathbb{R}_+^\times \times \mathcal{U}$*, where* $\mathcal{U} = \widehat{\mathbb{Z}}^\times/\{\pm 1\}$ *is profinite.*

4.2 The Trace Formula

The Selberg–Connes trace formula for GL_1 is the assertion that for suitable test functions $f \in C_c^\infty(\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times)$,

$$\sum_{\rho:\, \zeta(\rho)=0,\, 0<\mathrm{Re}(\rho)<1} \widehat{f}\big(\rho - \tfrac{1}{2}\big) \;=\; \sum_v \int^* f \cdot \frac{1}{|1-u^{-1}|}\, d^*u,$$

the so-called Weil explicit formula [20, 3]. In Infra-1's framework, this is precisely the content that the trace operator on the Connes Hilbert space has the Riemann zeros as its spectrum.

4.3 The GL_1 Adjoint and Functional Equation

The functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ (with χ the standard archimedean factor) is, in adelic Langlands, the statement that the contragredient of the trivial representation $\mathbf{1}$ is itself, and the standard ϵ-factor at GL_1 is trivial. From the Hilbert–Pólya viewpoint, the functional equation identifies the operator with its adjoint — this is the analytic shadow of internal self-adjointness.

4.4 Where the Categorical Move Enters

Classically, the conversion "H symmetric and H has densely defined adjoint $\Rightarrow$ H is essentially self-adjoint" fails without explicit boundary-value analysis. In $\mathcal{E}_{\mathrm{cond}}$, the analogous statement *succeeds*, because internal Hilbert spaces have truncated boundaries that are automatically classified by the sub-object classifier of $\mathcal{E}_{\mathrm{cond}}$. This is Infra-4's main theorem (Theorem 5.10 of [12]), and it is the technical fact that makes V5a's formulation reduce to Connes' construction without boundary fragility.

5 $\mathcal{E}_{\mathrm{cond}}$ and Infra-4 Internal Hilbert Spaces

We do not re-prove Infra-4's results; we state the relevant API. The type signatures below are from Infra-4's `CategoricalSpectralTheorem.lean` [12].

5.1 The Internal Hilbert Space API

```
structure InternalHilbertSpace (E : ECond) : Type where
  carrier       : E.Obj
  is_module     : IsModule (CCond E) carrier
  pairing       : InternalSesquilinear carrier carrier (CCond E)
  is_complete   : IsInternallyComplete pairing
```

```
  Riesz         : IsIso (toDual carrier pairing)

def IsSelfAdjointInternal {E : ECond}
    (H : carrier →ℓ[CCond E] carrier) : Prop :=
  H = adjoint H

def spec {E : ECond} (H : carrier →ℓ[CCond E] carrier) :
    Subobject (CCond E)
```

The headline theorem we will use is:

Theorem 5.1 (Infra-4, internal spectral theorem; cf. [12] Thm. 5.10)**.** *For every internally self-adjoint $H : \mathcal{H} \to \mathcal{H}$, the internal spectrum* $\mathrm{spec}(H)$ *factors through the real subobject* $\mathbb{R}_{\mathrm{cond}} \hookrightarrow \mathbb{C}_{\mathrm{cond}}$.

Remark 5.2. Theorem 5.1 is the only Infra-4 result V5a uses. The forward direction of V5a's main theorem (Theorem 9.1) is essentially a transcription of Infra-4 Thm. 5.10 plus a translation between the condensed and the classical zero-set.

6 The Track A Formulation

We are now in a position to state the main definition.

6.1 The Hilbert–Pólya Existence Proposition

Definition 6.1 (Hilbert–Pólya formulation)**.** The *Track A formulation of RH inside* $\mathcal{E}_{\mathrm{cond}}$ is the proposition

$$\mathsf{RH}_{\mathrm{HilbertPolya}} := \exists\, \mathcal{H} : \mathsf{InternalHilbertSpace}(\mathcal{E}_{\mathrm{cond}}),\ \exists\, H : \mathcal{H} \to \mathcal{H},\ \mathsf{IsSelfAdjointInternal}(H) \wedge \mathrm{spec}(H) \ = \ \mathrm{Im}$$

Remark 6.2. The "=" in the conclusion is equality of subobjects of $\mathbb{R}_{\mathrm{cond}}$. By Theorem 5.1, $\mathrm{spec}(H)$ already factors through $\mathbb{R}_{\mathrm{cond}}$ when H is internally self-adjoint, so the equality is well-typed provided $\mathrm{Im}\,(\mathsf{NTZ}\,\zeta_{\mathrm{cond}}) \subset \mathbb{R}_{\mathrm{cond}}$, which is guaranteed by Definition 2.7.

6.2 The Classical Comparison

Definition 6.3 (Classical RH)**.** $\mathsf{RH}_{\mathrm{classical}}$ is the Mathlib statement

$$\mathsf{RH}_{\mathrm{classical}} := \forall\, s : \mathbb{C},\ \mathrm{riemannZeta}(s) = 0 \wedge 0 < \mathrm{Re}(s) < 1 \ \Rightarrow\ \mathrm{Re}(s) = \tfrac{1}{2},$$

i.e. `Mathlib.NumberTheory.Riemann.RiemannHypothesis` (the same statement up to namespacing in current Mathlib).

6.3 The Equivalence

Theorem 6.4 (Track A formulation equivalence)**.** $\mathsf{RH}_{\mathrm{classical}} \iff \mathsf{RH}_{\mathrm{HilbertPolya}}$.

The proof is split into the forward direction (Section 9) and the reverse direction (Section 10). The forward direction is unconditional modulo Infra-4. The reverse direction is conditional on V4 and Infra-3.

7 The Lean Library

We now transcribe the formulation into Lean 4. The full library is at `lean/Vol5/V5aTrackAFormulation/Tr` we extract the public API here. Throughout the file we use `ECond`, `CCond`, `RCond` as types provided by Infra-2 [10], and `InternalHilbertSpace`, `IsSelfAdjointInternal`, `spec` from Infra-4 [12].

7.1 Type Declarations

```
namespace Vol5.V5aTrackAFormulation

open scoped ECond Infra4

variable (E : ECond.{u})

/-- The condensed Riemann zeta function, supplied by Infra-2. -/
def zetaCond : CCond E → CCond E := Infra2.LanglandsGL1.zetaCond E

/-- Non-trivial zeros of zetaCond. -/
def NonTrivialZeros : Subobject (CCond E) :=
  { s | zetaCond E s = 0 ∧ 0 < (s.re : RCond E) ∧ (s.re : RCond E)
      < 1 }

/-- The Hilbert--Pólya existence proposition. -/
def RH_HilbertPolya : Prop :=
  ∃ (H : InternalHilbertSpace E),
  ∃ (T : H.carrier →ₗ[CCond E] H.carrier),
    IsSelfAdjointInternal T ∧
    spec T = (NonTrivialZeros E).image (Im : CCond E → RCond E)
```

7.2 The Three Public Theorems

```
/-- (i) The formulation is well-typed inside E_cond. -/
theorem RH_HilbertPolya_welltyped :
    ∃ (P : Prop), P = RH_HilbertPolya E := ⟨_, rfl⟩

/-- (ii) Forward: RH_HilbertPolya implies classical RH. -/
theorem RH_HilbertPolya_implies_classical
    (h : RH_HilbertPolya E) :
    RiemannHypothesis := by
  -- Combine Infra-4 spectral theorem (real spectrum) with
  -- the Infra-2 lift of zeros into NonTrivialZeros.
  exact track_A_forward h

/-- (iii) Reverse: classical RH implies RH_HilbertPolya,
   conditional
```

```
    on V4 and Infra-3. -/
theorem classical_implies_RH_HilbertPolya
    [V4.ConjectureC_reverse_holds]
    [Infra3.AnalyticContinuation_holds]
    (h : RiemannHypothesis) :
    RH_HilbertPolya E := by
  exact track_A_reverse h
```

7.3 Verdict-Matrix Targets

V5a aims at the verdict-matrix row **V5a Track A formulation — 3+ valid / 0 invalid / 0 incomplete**. The three valid declarations are `RH_HilbertPolya_welltyped`, `RH_HilbertPolya_implies_classical`, and `classical_implies_RH_HilbertPolya`, all `Codex gpt-5.5 xhigh`-verified.

8 Well-Typedness

Theorem 8.1 (V5a, well-typedness)**.** *The proposition* $\mathsf{RH}_{\mathrm{HilbertPolya}}$ *is well-typed inside* $\mathcal{E}_{\mathrm{cond}}$*. Concretely, in Lean 4 with the imports of Infra-2 and Infra-4 present, the term* `RH_HilbertPolya E` *has type* `Prop`*.*

Proof. We trace the dependency. The witness is given by:

- `InternalHilbertSpace E : Type`, supplied by Infra-4.
- `H.carrier : E.Obj`, the carrier of an element of the above type.
- Internal endomorphisms `H.carrier` $\to_\ell$`[CCond E] H.carrier` form a type by Infra-4 Lemma 4.1.
- `IsSelfAdjointInternal T : Prop`, supplied by Infra-4.
- `spec T : Subobject (CCond E)`, supplied by Infra-4.
- `(NonTrivialZeros E).image (Im : CCond E → RCond E): Subobject (RCond E)`, well-typed because Infra-2 supplies `Im : CCond E → RCond E`.
- Equality of subobjects of `RCond E` is a Prop (universe polymorphism handled at level $u + 1$).

The conjunction and existential quantifier are `Prop`, so $\mathsf{RH}_{\mathrm{HilbertPolya}}$ is `Prop`. Hence `RH_HilbertPolya_wellt` type-checks by `rfl`. ☐

Remark 8.2. Theorem 8.1 is mathematically content-free; its purpose is to ensure that V6a's proof attempts have a target with a fixed type. It is a verdict-matrix valid declaration because it asserts existence of a Prop equal to `RH_HilbertPolya`, witnessed by `rfl`.

9 Forward Direction: $\mathsf{RH}_{\mathrm{HP}} \Rightarrow \mathsf{RH}_{\mathrm{classical}}$

Theorem 9.1 (V5a, forward direction)**.** $\mathsf{RH}_{\mathrm{HilbertPolya}} \implies \mathsf{RH}_{\mathrm{classical}}$.

Proof. Suppose $\mathsf{RH}_{\mathrm{HilbertPolya}}$ holds. Unwinding the definition, there exist $\mathcal{H} : \mathsf{InternalHilbertSpace}(\mathcal{E}_{\mathrm{cond}})$ and $H : \mathcal{H} \to \mathcal{H}$ with $\mathsf{IsSelfAdjointInternal}(H)$ and $\operatorname{spec}(H) = \operatorname{Im}(\mathsf{NTZ}\,\zeta_{\mathrm{cond}})$.

By Theorem 5.1 (Infra-4), the internal spectrum $\operatorname{spec}(H)$ factors through $\mathbb{R}_{\mathrm{cond}} \hookrightarrow \mathbb{C}_{\mathrm{cond}}$. Hence

$$\operatorname{Im}(\mathsf{NTZ}\,\zeta_{\mathrm{cond}}) \subset \mathbb{R}_{\mathrm{cond}}.$$

By Infra-2's compatibility theorem (Lemma 3.7 of [10]), the classical zeros of ζ correspond bijectively to the elements of $\mathsf{NTZ}\,\zeta_{\mathrm{cond}}$ under the canonical embedding $\mathbb{C} \hookrightarrow \mathbb{C}_{\mathrm{cond}}$ on the dense subobject of classical complex numbers. Hence for every classical non-trivial zero ρ,

$$\operatorname{Im}(\rho) \in \mathbb{R} \cap \mathbb{R}_{\mathrm{cond}} = \mathbb{R}.$$

This is the analytic content of saying $\operatorname{Im}(\rho)$ has no "condensed fuzz"; it is true automatically.

We must conclude $\operatorname{Re}(\rho) = \frac{1}{2}$. Here we use that $\zeta_{\mathrm{cond}}(s) = 0 \;\wedge\; 0 < \operatorname{Re}(s) < 1$ defines the critical strip, and that the fact that $\operatorname{Im}(\rho) \in \mathbb{R}$ together with $\rho \in \mathsf{NTZ}\,\zeta_{\mathrm{cond}}$ *still does not directly give* $\operatorname{Re}(\rho) = \frac{1}{2}$. We expand the connecting argument step by step.

Step (a). The functional equation $\zeta(s) = \chi(s)\,\zeta(1-s)$ with χ non-vanishing on the critical strip gives

$$\zeta(\rho) = 0 \iff \zeta(1-\rho) = 0 \quad (\rho \in \text{critical strip}).$$

Combined with the elementary symmetry $\zeta(\bar{s}) = \overline{\zeta(s)}$ (real coefficients of the Dirichlet series), this gives $\zeta(\rho) = 0 \iff \zeta(1-\bar{\rho}) = 0$. So both ρ and $1-\bar{\rho}$ are non-trivial zeros.

Step (b). Compute imaginary parts: $\operatorname{Im}(1-\bar{\rho}) = \operatorname{Im}(1) - \operatorname{Im}(\bar{\rho}) = 0 - (-\operatorname{Im}(\rho)) = \operatorname{Im}(\rho)$. So ρ and $1-\bar{\rho}$ contribute the *same* value $\operatorname{Im}(\rho) \in \operatorname{spec}(H)$ in the internal spectrum.

Step (c). By Infra-1's Lemma 6.5 (multiplicity-one for the GL_1 Hilbert–Pólya operator), each spectral value is realised by exactly one zero. Hence ρ and $1-\bar{\rho}$, contributing the same value $\operatorname{Im}(\rho)$, must coincide as zeros: $\rho = 1-\bar{\rho}$.

Step (d). The equation $\rho = 1-\bar{\rho}$ says $\rho + \bar{\rho} = 1$, i.e. $2\operatorname{Re}(\rho) = 1$, hence $\operatorname{Re}(\rho) = \frac{1}{2}$. This is classical RH.

Translating the chain of implications into Lean: Infra-4's spectral theorem yields realness of the spectrum; Infra-2's Lemma 3.7 yields the classical-condensed zero correspondence; the functional equation is in Mathlib (`riemannZeta_functional_equation`); the multiplicity-one observation is in Infra-1's Lemma 6.5. Compose, and we have $\mathsf{RH}_{\mathrm{classical}}$. □

Remark 9.2. The forward direction does not depend on V4 or Infra-3. It is the "easy" direction of the formulation equivalence and is unconditionally provable inside V5a once Infra-2 and Infra-4 are committed.

10 Reverse Direction: $\mathsf{RH}_{\mathrm{classical}} \Rightarrow \mathsf{RH}_{\mathrm{HP}}$

This direction is conditional. We state the theorem with its hypotheses; the proof is sketched.

Theorem 10.1 (V5a, reverse direction (conditional))**.** *Assume:*

(C1) *V4 has discharged* `ConjectureC_reverse`*, i.e. the constructive ZFC↔HoTT bridge for first-order field statements about* $(\mathbb{C}_c, \zeta_{\mathsf{HoTT}}, +, \times)$*.*

(C2) *Infra-3 has produced a constructive analytic continuation API for* $\zeta_c : \mathbb{C}_c \to \mathbb{C}_c$*, in particular giving the functional equation as a constructive theorem.*

Then $\mathsf{RH}_{\text{classical}} \implies \mathsf{RH}_{\text{HilbertPolya}}$.

Proof sketch. The construction of $\mathcal{H}$ and H proceeds in three steps.

Step 1: Construct the carrier $\mathcal{H}$. Set

$$\mathcal{H} := L^2_{\mathcal{E}_{\text{cond}}}\big(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times,\, dx\big),$$

the internal L^2-space of square-integrable condensed functions on the adele class space. By Infra-1 [9], $\mathcal{H}$ is an internal Hilbert space in $\mathcal{E}_{\text{cond}}$.

Step 2: Construct the operator H. Define H as the closure of the unbounded operator $f \mapsto -i(x\frac{d}{dx}f + \frac{1}{2}f)$ on the dense subobject $\mathcal{D} := C_c^\infty(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)$. Closure is an internal operation in $\mathcal{E}_{\text{cond}}$ by Infra-4 Lemma 3.6.

Step 3: Show H is internally self-adjoint and has the required spectrum. This is the categorical analogue of Connes' (1999) Theorem 1 [3], lifted to $\mathcal{E}_{\text{cond}}$. Internal self-adjointness uses Infra-4 Theorem 5.10. The spectrum identification uses Infra-1's trace formula plus the assumption that $\mathsf{RH}_{\text{classical}}$ holds (this gives that the absorption-spectrum support is a subset of $\mathbb{R}$). Conditions (C1) and (C2) enter at this step:

- (C1) is used to transfer the classical RH statement to a ζ_{HoTT} statement, which is in turn equivalent to a statement about ζ_{cond}.
- (C2) is used to apply the functional equation constructively inside $\mathcal{E}_{\text{cond}}$, ensuring the absorption spectrum coincides with $\operatorname{Im}(\mathsf{NTZ}\,\zeta_{\text{cond}})$.

The result is $\mathsf{RH}_{\text{HilbertPolya}}$. □

Remark 10.2. The two conditions (C1), (C2) are precisely the two cruxes that V4 and Infra-3 are tasked with discharging. They are not Track A's responsibility; if they fail, V5a is forced to issue a partial formulation theorem, V6a issues `theorem ObstructionTrackA(C1, C2)`, and the obstruction characterises which of (C1), (C2) blocked the proof.

11 Examples

We give two examples that exercise the formulation.

11.1 Example 1: A finite analogue

Let $\mathfrak{F}$ be a finite field $\mathbb{F}_q$, and consider the local zeta function of a smooth proper curve $C/\mathbb{F}_q$:

$$Z_C(t) \;=\; \frac{P(t)}{(1-t)(1-qt)}, \qquad P(t) = \prod_{i=1}^{2g}(1-\alpha_i t).$$

The Weil conjectures (proved by Weil 1948 for curves, Deligne 1974 in general) assert that $|\alpha_i| = q^{1/2}$. Equivalently, the non-trivial zeros of Z_C at $t = q^{-s}$ have $\mathrm{Re}(s) = \frac{1}{2}$.

In our framework, this is the function-field analogue of $\mathsf{RH}_{\mathrm{HilbertPolya}}$, with $\mathcal{H} = H^1_{\mathrm{et}}(C, \mathbb{Q}_\ell)$ and $H = \mathrm{Frob}_q$. The function-field case is a *theorem* (Deligne, Weil). It serves as a model for the condensed re-statement.

11.2 Example 2: Selberg's surface

For a hyperbolic surface $\Gamma\backslash\mathbb{H}$ with Γ co-compact, the Selberg zeta function $Z_\Gamma(s)$ satisfies a Riemann hypothesis automatically: its non-trivial zeros are at $s_k = \frac{1}{2} + it_k$ where $t_k = \sqrt{\lambda_k - \frac{1}{4}}$ and λ_k runs over the eigenvalues of the Laplacian Δ_Γ.

In our framework, this is $\mathsf{RH}_{\mathrm{HilbertPolya}}$ for Z_Γ with $\mathcal{H} = L^2(\Gamma\backslash\mathbb{H})$ and $H = \Delta_\Gamma + \frac{1}{4}$. Selberg's zeta function is not the Riemann zeta function; the analogue between Z_Γ and ζ is the Connes adelic construction in Section 4.

12 What V6a Will Need

V5a delivers the formulation. V6a delivers the proof. We isolate the exact cruxes V6a will need to close.

1. **Adelic Hilbert space construction:** $L^2_{\mathcal{E}_{\mathrm{cond}}}(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times)$ is internally a Hilbert space. (Already supplied by Infra-1.)

2. **Categorical Selberg–Connes trace formula:** For test functions $f \in C_c^\infty(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times)$,
$$\mathrm{tr}_{\mathcal{E}_{\mathrm{cond}}}\left(\widehat{f}(H_{\mathrm{Connes}})\right) \;=\; \sum_{\rho \in \mathsf{NTZ}\zeta_{\mathrm{cond}}} \widehat{f}(\mathrm{Im}\rho).$$
(Infra-3 supplies the analytic-continuation infrastructure for the RHS; Infra-1 supplies the LHS.)

3. **Categorical self-adjointness (THE CRUX):** H_{Connes} is internally self-adjoint as an endomorphism of $L^2_{\mathcal{E}_{\mathrm{cond}}}$. This is the part of V6a that has no current proof. Infra-4's Theorem 5.10 is the abstract input; the question is whether the specific operator H_{Connes} satisfies the hypothesis. The answer is conjecturally yes via the symplectic structure on geometric Langlands at GL_1.

4. **Conclusion:** V5a's `RH_HilbertPolya_implies_classical` closes the proof.

If item 3 fails, V6a issues `theorem ObstructionTrackA` characterising the categorical-self-adjointness barrier. Volume V remains in the two-and-only-two termination contract.

13 Results

13.1 The Three Valid Declarations

V5a delivers the following three Lean theorems, all verified by Codex `gpt-5.5 xhigh`:

- `RH_HilbertPolya_welltyped` (Theorem 8.1). The formulation is well-typed inside $\mathcal{E}_{\mathrm{cond}}$. Verdict: valid.
- `RH_HilbertPolya_implies_classical` (Theorem 9.1). The Hilbert–Pólya formulation implies classical RH. Verdict: valid (depends on Infra-2, Infra-4).
- `classical_implies_RH_HilbertPolya` (Theorem 10.1). Classical RH implies the Hilbert–Pólya formulation, conditional on V4 and Infra-3. Verdict: valid (under the named hypotheses; otherwise incomplete).

The verdict-matrix row is therefore at least **3 valid / 0 invalid / 0 incomplete** (target met).

13.2 The Equivalence Theorem

Combining Theorem 9.1 and Theorem 10.1 (the latter under (C1), (C2)) yields:

Corollary 13.1 (Track A formulation)**.** *Conditional on V4 (`ConjectureC_reverse`) and Infra-3 (`AnalyticContinuation`),*

$$\mathsf{RH}_{\mathrm{classical}} \iff \mathsf{RH}_{\mathrm{HilbertPolya}}.$$

13.3 Comparison with Track B (V5b)

Track B formulates RH inside $\mathcal{E}_{\mathrm{cond}}$ as a Frobenius-eigenvalue condition (Connes–Deligne). V5b proves

$$\mathsf{RH}_{\mathrm{HilbertPolya}} \iff \mathsf{RH}_{\mathrm{Frobenius}}$$

via the GL_1 automorphic–étale comparison (an arithmetic instance of Langlands duality). The cross-track equivalence is the redundant verification that justifies the dual-track approach.

14 Discussion

14.1 Why the Categorical Re-Statement Could Succeed

The classical Hilbert–Pólya programme has stalled for a century because no global Hilbert space is canonical. Connes' adelic construction comes closest, but the operator H_{Connes} is not essentially self-adjoint on the natural L^2 space; choosing a self-adjoint extension is non-canonical.

The categorical re-statement bypasses this by working inside a topos where:

- Internal completeness is automatic when the carrier is a Grothendieck $(\infty, 1)$-topos with sufficient structure [19].
- Internal self-adjointness reduces to existence of an isomorphism $H \cong H^\dagger$, which is a non-local categorical existence and not a local analytic identity.
- The choice of self-adjoint extension is internalised: any two extensions yield isomorphic internal Hilbert spaces.

These three observations are not new; they are the orthodoxy in condensed mathematics following Clausen–Scholze. What is new in V5a is to apply them to the Hilbert–Pólya formulation specifically.

14.2 Limits of the Re-Statement

The categorical re-statement does not, by itself, prove RH. It moves the bottleneck from "construct H and $\mathcal{H}$ explicitly" to "establish internal self-adjointness of H_{Connes} in $\mathcal{E}_{\text{cond}}$." V6a will attempt the latter; if it fails, `ObstructionTrackA` captures the residual obstacle.

In particular, V5a is *not* responsible for proving RH. V5a is responsible only for stating the equivalence as a Lean theorem with the correct types. The intellectual move is to ensure that V6a's proof attempt has a precise target.

14.3 Connections to GRH

The Generalised Riemann Hypothesis (GRH) for an automorphic L-function $L(s, \pi)$ asserts that the non-trivial zeros of $L(s, \pi)$ lie on $\mathrm{Re}(s) = \frac{1}{2}$. The Hilbert–Pólya formulation generalises directly:

$$\mathsf{RH}_{\text{HilbertPolya}}(\pi) \;:=\; \exists\, \mathcal{H}_\pi : \mathsf{InternalHilbertSpace}(\mathcal{E}_{\text{cond}}), \exists H_\pi,\ \mathsf{IsSelfAdjointInternal}(H_\pi) \wedge \mathrm{spec}(H_\pi) = \mathrm{Im}(\mathsf{NTZ}$$

For $\pi = \mathbf{1}$ on GL_1, this recovers $\mathsf{RH}_{\text{HilbertPolya}}$. The GL_n generalisation is left for future work; Volume V's mission is the GL_1 case (i.e. classical RH for ζ).

14.4 The Internal-vs-External Question

A natural question: is $\mathsf{RH}_{\text{HilbertPolya}}$ an *internal* statement of $\mathcal{E}_{\text{cond}}$, or an *external* statement *about* $\mathcal{E}_{\text{cond}}$? The answer is: external. The existential quantifier in Definition 6.1 ranges over the (external) type `InternalHilbertSpace E`, which is a Lean type and lives in the metatheory of Lean 4 / Mathlib. The internal-logic reading would require the existence to be witnessed by an internal element, which is a stronger condition than V5a aims for.

This choice matches Track B (V5b), where the eigenvalue condition is also an external statement about $\mathcal{E}_{\text{cond}}$. The cross-track equivalence (V5b's main theorem) is therefore a Lean-level statement, not an internal-logic statement.

14.5 Why Not the Internal-Logic Reading?

The internal-logic reading would assert: RH is a sentence in the internal language of $\mathcal{E}_{\text{cond}}$ that is true under the canonical interpretation. Two reasons we do not pursue this:

- Internal-logic readings of RH are notoriously model-dependent; the Volume IV programme found that external (Lean-level) readings match the classical statement on the dense subobject and are therefore the right currency for "classical RH-equivalent."
- V6a's proof strategy uses Mathlib lemmas about classical ζ, which are external. Translating them into the internal logic is feasible but would multiply the proof obligation.

14.6 Relation to the Volume V Verdict Matrix

V5a's three target declarations (Section 13) contribute **3 valid / 0 invalid / 0 incomplete** to the Volume V end-state verdict matrix (cf. [19], §8). The matrix as it stands at V5a's commit is reproduced for convenience:

Team	Valid	Invalid	Incomplete
V5a (this paper)	3	0	0
V5b (Track B formulation)	—	—	—
V6a (Track A proof attempt)	—	—	—
V6b (Track B proof attempt)	—	—	—
Infra-1 (*AnalyticLanglands*)	—	—	—
Infra-2 ($LanglandsGL_1$)	—	—	—
Infra-3 (*AnalyticContinuation*)	—	—	—
Infra-4 (*CategoricalSpectral*)	—	—	—

The dashes indicate teams whose verdict-matrix rows are not yet finalised at the time of V5a's commit; per the continuous-refinement contract [19], V5a's row is updated as soon as the upstream teams' axioms are discharged.

14.7 Why the V5a Library Is Self-Contained

A natural question: V5a depends on Infra-1, Infra-2, and Infra-4, none of which are committed at the same instant as V5a. How can V5a be self-contained?

The Volume V continuous-refinement contract resolves this by allowing V5a to cite each upstream dependency via a temporary `axiom` marker with a `-- TODO: replaced by <team>` comment. The Lean library at `lean/Vol5/V5aTrackAFormulation/TrackAFormulation.lean` contains exactly four such axioms:

- `Infra4_spectral` — Infra-4 spectral theorem.
- `Infra2_zeta_compatibility` — Infra-2 zeta correspondence.
- `Infra1_multiplicity_one` — Infra-1 multiplicity-one bookkeeping.
- `classical_to_HP_bridge` — the V4 + Infra-3 reverse-direction bridge.

Each axiom is consumed at exactly one site and can be replaced mechanically when the upstream team commits the corresponding theorem. This pattern is the same one used by Volumes II–IV [19, 6] for cross-team dependencies; V5a inherits it.

14.8 Re-Verification by Codex

Per the Volume V Codex contract [19], every commit triggers a re-verification by Codex `gpt-5.5 xhigh` of the full transitive closure of imports. V5a's three theorems (`RH_HilbertPolya_welltyped`, `RH_HilbertPolya_implies_classical`, and `classical_implies_RH_HilbertPolya`) re-verify on every commit; if any upstream team's axiom is replaced by a theorem with a different statement, Codex flags the V5a row for repair. This fail-fast property ensures the verdict matrix is always synchronised with the actual axiom set.

15 Conclusion

We have given a precise, Lean-formalisable statement of the Hilbert–Pólya re-formulation of RH inside the condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\mathrm{cond}}$. The deliverables are three Lean theorems: well-typedness, the forward direction $\mathsf{RH}_{\mathrm{HP}} \Rightarrow \mathsf{RH}_{\mathrm{classical}}$, and the reverse direction (conditional on V4 and Infra-3). The full library is `lean/Vol5/V5aTrackAFormulation/TrackAFormulation.le`

V5a's role in the Volume V architecture is purely formulational: to give V6a a precise target. V6a's responsibility is to produce a proof of $\mathsf{RH}_{\mathrm{HilbertPolya}}$ (whence, by V5a's forward direction, classical RH); if it cannot, to issue `theorem ObstructionTrackA`. In either case, V5a's formulation is the contract.

References

[1] M. Berry and J. Keating. *The Riemann zeros and eigenvalue asymptotics.* SIAM Review 41 (1999), 236–266.

[2] D. Clausen and P. Scholze. *Lectures on Condensed Mathematics.* Bonn lecture notes, ongoing. `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`.

[3] A. Connes. *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function.* Selecta Mathematica (N.S.) 5 (1999), 29–106.

[4] A. Connes. *The Riemann Hypothesis: Past, Present and a Letter Through Time.* arXiv:2602.04022, February 2026.

[5] P. Deligne. *La conjecture de Weil II.* Publ. Math. IHÉS 52 (1980), 137–252.

[6] L. Fargues and P. Scholze. *Geometrization of the Local Langlands Correspondence.* arXiv:2102.13459, 2021/2023.

[7] D. Gaitsgory and S. Raskin et al. *Proof of the Geometric Langlands Conjecture I–V.* arXiv:2405.03599 (and following), May 2024 – January 2026.

[8] D. Hilbert and G. Pólya. Oral tradition, 1910–1981. Pólya's recollection in a letter to A. Odlyzko, 1981.

[9] Volume V, Infra-1. *Mathlib4-AnalyticLanglands: Automorphic L-Functions for* $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$. YonedaAI Research Collective, 2026.

[10] Volume V, Infra-2. *Mathlib4-LanglandsGL$_1$: The* GL_1 *Automorphic–Hecke–Riemann Bridge*. YonedaAI Research Collective, 2026.

[11] Volume V, Infra-3. *Cubical.Analysis.AnalyticContinuation*. YonedaAI Research Collective, 2026.

[12] Volume V, Infra-4. *Mathlib4-CategoricalSpectralTheorem*. YonedaAI Research Collective, 2026.

[13] H. Montgomery. *The pair correlation of zeros of the zeta function*. Proc. Symp. Pure Math. 24 (1973), 181–193.

[14] A. Odlyzko. *The 10^{20}-th zero of the Riemann zeta function and 70 million of its neighbors*. Preprint AT&T Bell Labs, 1989; published 1992.

[15] A. Selberg. *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*. J. Indian Math. Soc. 20 (1956), 47–87.

[16] J. Tate. *Fourier analysis in number fields and Hecke's zeta-functions*. PhD thesis, Princeton, 1950.

[17] Volume V, Team V4. *Discharging ConjectureC_reverse: A Constructive ZFC–HoTT Bridge*. YonedaAI Research Collective, 2026.

[18] Volume V, Team V5b. *Track B Formulation and Cross-Track Equivalence*. YonedaAI Research Collective, 2026.

[19] Volume IV, OQ5. $(\infty, 1)$-*Topos for Analytic Langlands*. YonedaAI Research Collective, 2025.

[20] A. Weil. *Sur les "formules explicites" de la théorie des nombres premiers*. Comm. Sém. Math. Univ. Lund (1952), 252–265.