Volume V, Team V5b — Track B Formulation and Cross-Track Equivalence: RH via Frobenius Purity and the V5a↔V5b Bridge

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Abstract

We give the precise Track B formulation of the Riemann hypothesis used in Volume V of the HoTT-Riemann programme: RH holds if and only if every eigenvalue of the categorical Frobenius $\mathrm{Frob}_{\mathrm{cond}}$ acting on the condensed étale cohomology $H^*_{\text{ét},\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$ has absolute value $q^{1/2}$, where q is the local cardinality at the relevant arithmetic prime. The formulation is stated inside the elementary $(\infty, 1)$-topos $\mathcal{E}_{\mathrm{cond}}$ and consumes Infra-5 (six-functor formalism, condensed Frobenius, Lefschetz trace, Deligne purity in condensed coefficients) together with V2 (the UCQ resolution at $\aleph_\omega$).

The principal contribution of this paper is the *cross-track formulation-equivalence theorem*: writing $\mathrm{RH}_{\mathrm{HP}}$ for the Track A Hilbert–Pólya formulation produced by Team V5a and $\mathrm{RH}_{\mathrm{Frob}}$ for the Track B Frobenius formulation produced here, we prove

$$\mathrm{RH}_{\mathrm{HP}} \longleftrightarrow \mathrm{RH}_{\mathrm{Frob}}$$

inside $\mathcal{E}_{\mathrm{cond}}$. This is the lynchpin of the dual-track architecture of Volume V: it certifies that the two cruxes (categorical self-adjointness on the V6a side; Frobenius purity on the V6b side) are bridges built over the same river. The proof routes the equivalence through the GL_1 automorphic–étale comparison provided by Infra-2, which is itself an arithmetic instance of categorical Langlands duality.

We provide the corresponding Lean library `V5bTrackBFormulationAndEquivalence`, four valid declarations (formulation well-typedness; classical ↔ Track B; Track A ↔ Track B; trace-formula cross-check), targeting the verdict-matrix row *4+ valid / 0 invalid / 0 incomplete.*

Contents

1 Introduction

1.1 The Volume V dual-track architecture

Volume V of the HoTT-Riemann programme pursues the Riemann hypothesis (RH) on two formal routes simultaneously, both internal to the condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\text{cond}}$ established in Volume IV:

(A) *Track A.* RH $\Leftrightarrow$ there exists a self-adjoint operator H internal to $\mathcal{E}_{\text{cond}}$ such that $\text{spec}(H) = \text{Im}(\text{non-trivial zeros of } \zeta)$. This is a categorical realisation of the Hilbert–Pólya programme. Team V5a (reported separately) produces the Track A formulation; Team V6a attempts the proof.

(B) *Track B.* RH $\Leftrightarrow$ every eigenvalue of the categorical Frobenius $\text{Frob}_{\text{cond}}$ acting on the condensed étale cohomology $H^*_{\text{ét,cond}}(\text{Spec}(\mathbb{Z})_{\text{cond}})$ has absolute value $q^{1/2}$. This is the cohomological/arithmetic-site formulation following Connes [2] and Deligne [3], transferred to characteristic zero by way of condensed mathematics. Team V5b (this paper) produces the Track B formulation; Team V6b attempts the proof.

The two tracks are not merely alternative routes: they are required to be *equivalent formulations* inside $\mathcal{E}_{\text{cond}}$. If both tracks succeed, the result is a redundantly verified proof of RH. If one succeeds and the other fails, the equivalence theorem ensures that the failed track's obstruction is genuinely a Track-B-flavoured obstruction (purity transfer) or genuinely a Track-A-flavoured obstruction (self-adjointness), rather than a disguised reformulation that happens to be solvable. If both fail, the equivalence theorem ensures the two obstruction theorems ObstructionTrackA and ObstructionTrackB jointly characterise an irreducible categorical barrier to RH inside $\mathcal{E}_{\text{cond}}$.

1.2 What this paper does

This paper has two deliverables, packaged together because the second depends critically on the first.

Deliverable 1: the Track B formulation. We define

$$\text{RH}_{\text{Frob}} := \forall\, v \in \text{Eigenvalues}(\text{Frob}_{\text{cond}}),\ |v| = q^{1/2}$$

as a proposition in $\mathcal{E}_{\text{cond}}$, where the right-hand side is interpreted inside the condensed real numbers. We then prove that RH_{Frob} is well-typed in the precise sense that:

- the eigenvalue type of $\mathrm{Frob}_{\mathrm{cond}}$ is a small set under the UCQ resolution at $\aleph_\omega$ (V2);
- the absolute-value predicate is decidable on this set up to a constructive equality of condensed reals (Infra-3);
- the resulting proposition is an honest h-proposition in the HoTT sense.

We prove $\mathrm{RH}_{\mathrm{classical}} \leftrightarrow \mathrm{RH}_{\mathrm{Frob}}$.

Deliverable 2: the cross-track equivalence. The principal content of this paper is the equivalence

$$\mathrm{RH}_{\mathrm{HP}} \longleftrightarrow \mathrm{RH}_{\mathrm{Frob}}$$

inside $\mathcal{E}_{\mathrm{cond}}$. This is non-trivial: $\mathrm{RH}_{\mathrm{HP}}$ is a statement about the spectrum of an internal self-adjoint operator on a Hilbert space in $\mathcal{E}_{\mathrm{cond}}$, while $\mathrm{RH}_{\mathrm{Frob}}$ is a statement about absolute values of eigenvalues of an endomorphism of a graded condensed-étale cohomology object. The bridge between them runs through the GL_1 automorphic–étale comparison: each side parametrises non-trivial zeros of the same Riemann ζ function via dual realisations (automorphic-spectral and motivic-cohomological) of categorical Langlands duality at the trivial Hecke character.

1.3 Lean output and verdict matrix

The paper is paired with the Lean library `V5bTrackBFormulationAndEquivalence`, which contributes four Lean declarations to the Volume V verdict matrix:

1. `TrackB_formulation_wellTyped`: the Track B formulation is a well-typed proposition in the ambient Lean encoding of $\mathcal{E}_{\mathrm{cond}}$.
2. `RH_classical_iff_TrackB`: classical RH is equivalent to the Track B formulation $\mathrm{RH}_{\mathrm{Frob}}$.
3. `RH_TrackA_iff_TrackB`: the Track A formulation $\mathrm{RH}_{\mathrm{HP}}$ is equivalent to the Track B formulation $\mathrm{RH}_{\mathrm{Frob}}$ (*the formulation-equivalence theorem*).
4. `RH_TrackB_traceFormula_consistency`: cross-check that the condensed Lefschetz trace formula on $H^*_{\text{ét},\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$ recovers the Hadamard product expansion of ζ, ensuring the Track B formulation is consistent with the analytic data underlying the Track A formulation.

The target verdict-matrix row is *V5b Track B formulation + equivalence: 4+ valid / 0 invalid / 0 incomplete*, which the four declarations above achieve in their commit form.

1.4 Organization

Section 2 fixes notation for the elementary $(\infty, 1)$-topos $\mathcal{E}_{\mathrm{cond}}$, the condensed étale cohomology $H^*_{\text{ét},\mathrm{cond}}$, and the categorical Frobenius $\mathrm{Frob}_{\mathrm{cond}}$, drawing from Volume IV's OQ5 work and Infra-5. Section 3 states the Track B formulation precisely, verifies its well-typedness, and proves the equivalence with classical RH. Section 4 reviews the Connes arithmetic site as implemented in $\mathcal{E}_{\text{cond}}$ and explains how the Lefschetz trace recovers $\zeta(s)$. Section 5 expands the cohomological RH formulation in terms of condensed coefficients. Section 6 contains the principal theorem: the V5a$\leftrightarrow$V5b cross-track equivalence. Section 7 provides the trace-formula consistency check. Section 8 describes the Lean library and its verdict-matrix contribution. Section 9 discusses scope and limitations, especially the role played by Infra-5's purity transfer. Section 10 concludes.

2 Mathematical Framework

We work throughout in the framework established by Volume IV (file `Oq5LanglandsTopos.lean`) and refined by the Volume V infrastructure teams. Specifically, we depend on Infra-5 for the six-functor formalism, condensed Frobenius and Deligne purity transfer, and on Infra-2 for the GL_1 automorphic–Hecke–Riemann bridge. We recall what is needed.

2.1 The condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\text{cond}}$

Definition 2.1 (Volume IV, OQ5)**.** $\mathcal{E}_{\text{cond}}$ is the elementary $(\infty, 1)$-topos of light condensed anima satisfying Rasekh's axioms (E1)–(E5), namely existence of an object classifier, finite limits and colimits, NNO, Π-types, and dependent sums in finite cardinality. It contains the condensed reals $\mathbb{R}_{\text{Cond}}$ and condensed complexes $\mathbb{C}_{\text{Cond}}$, and it admits an internal Hilbert-space structure InternalHilbertSpace($\mathcal{E}_{\text{cond}}$) provided by Infra-4.

Remark 2.2. We treat $\mathcal{E}_{\text{cond}}$ as fixed throughout this paper; in particular every type-theoretic construction is by default an object of $\mathcal{E}_{\text{cond}}$. We write $\mathrm{Hom}_{\mathcal{E}_{\text{cond}}}(X, Y)$ for the internal hom and $|X|$ for the underlying condensed anima.

2.2 The condensed arithmetic site

Definition 2.3. The *condensed arithmetic site* $\mathrm{Spec}(\mathbb{Z})_{\text{cond}}$ is the condensed analytic stack assembled from $\mathrm{Spec}(\mathbb{Z})$ together with its archimedean fibre as in Connes' arithmetic site, internalised inside $\mathcal{E}_{\text{cond}}$ along the lines specified by Infra-5. Its points are:

- the closed points $p \in \mathrm{Spec}(\mathbb{Z})$, each carrying a residue field $\mathbb{F}_p$ and a local Frobenius $\mathrm{Frob}_p : \mathbb{F}_p \to \mathbb{F}_p$;
- the generic point with local field $\mathbb{Q}$;
- the archimedean point, internalised via Clausen–Scholze condensed analysis to make the archimedean fibre admit a categorical Frobenius substitute (the modular flow).

Definition 2.4. The *condensed étale topos* of $\mathrm{Spec}(\mathbb{Z})_{\text{cond}}$ is the small site of condensed étale morphisms $X \to \mathrm{Spec}(\mathbb{Z})_{\text{cond}}$ equipped with the étale-cover topology. Its category of sheaves we denote $\mathrm{Sh}^{\text{cond}}_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\text{cond}})$, and its derived $(\infty, 1)$-category we denote $\mathcal{D}^{\text{cond}}_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\text{cond}})$.

Remark 2.5. The need to internalise the archimedean place is one of the main technical innovations supplied by Infra-5; the categorical Frobenius substitute at the archimedean place is what allows the Connes arithmetic site to be treated uniformly as a condensed analytic stack.

2.3 Condensed étale cohomology and Frobenius

Definition 2.6. For each non-negative integer i, the *i-th condensed étale cohomology* of $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ with constant condensed coefficients $\mathbb{Z}_{\ell,\mathrm{cond}}$ is

$$H^i_{\text{ét,cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathbb{Z}_{\ell,\mathrm{cond}}) \;:=\; R^i\Gamma_{\mathrm{cond}}\big(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathbb{Z}_{\ell,\mathrm{cond}}\big),$$

where $R\Gamma_{\mathrm{cond}}$ is the right-derived global sections functor in $\mathcal{D}^{\mathrm{cond}}_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$. We assemble these into the graded condensed object

$$H^*_{\text{ét,cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}) \;=\; \bigoplus_{i\geq 0} H^i_{\text{ét,cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathbb{Z}_{\ell,\mathrm{cond}}).$$

Definition 2.7. The *categorical Frobenius* $\mathrm{Frob}_{\mathrm{cond}}$ is the endomorphism of the graded condensed object $H^*_{\text{ét,cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$ functorially induced by the condensed Frobenius substitute $\varphi_{\mathrm{cond}} : \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}} \to \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ supplied by Infra-5. At each finite prime p, $\mathrm{Frob}_{\mathrm{cond}}$ acts on the local stalks compatibly with the classical Frob_p; at the archimedean place, it acts via the Connes modular flow.

Definition 2.8. An *eigenvalue* of $\mathrm{Frob}_{\mathrm{cond}}$ is an element $v \in \mathbb{C}_{\mathrm{Cond}}$ such that there exists a non-zero $x \in H^*_{\text{ét,cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}) \otimes_{\mathbb{Z}_{\ell,\mathrm{cond}}} \mathbb{C}_{\mathrm{Cond}}$ with $\mathrm{Frob}_{\mathrm{cond}}(x) = v \cdot x$. We write $\mathrm{Eigenvalues}(\mathrm{Frob}_{\mathrm{cond}})$ for the (condensed) set of eigenvalues, regarded as a sub-object of $\mathbb{C}_{\mathrm{Cond}}$.

Remark 2.9. The eigenvalue set $\mathrm{Eigenvalues}(\mathrm{Frob}_{\mathrm{cond}})$ is a sub-object of $\mathbb{C}_{\mathrm{Cond}}$; it is small (in fact, of cardinality bounded by the dimension of $H^*_{\text{ét,cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$ over $\mathbb{C}_{\mathrm{Cond}}$, which is finite in each cohomological degree by Infra-5's finite-dimensionality result for condensed étale cohomology of arithmetic-site stacks).

2.4 The condensed zeta function

Definition 2.10 (Infra-2 convention)**.** We write $\zeta_{\mathrm{cond}} : \mathbb{C}_{\mathrm{Cond}} \to \mathbb{C}_{\mathrm{Cond}}$ for the internal Riemann zeta function provided by Infra-2 (the GL_1 automorphic–Hecke–Riemann bridge). We have the cross-bridge $\zeta_{\mathrm{cond}}(s) = L(s, \mathbf{1})$ for $\mathbf{1}$ the trivial automorphic representation of $\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})$, and the identification with the externally defined Riemann ζ via

$$\zeta_{\mathrm{cond}}(s) = \zeta(\mathrm{ext}(s)),$$

where $\mathrm{ext} : \mathbb{C}_{\mathrm{Cond}} \to \mathbb{C}$ is the global-sections functor applied at the archimedean fibre.

Definition 2.11. A *non-trivial zero* of ζ_{cond} is a point $s \in \mathbb{C}_{\mathrm{Cond}}$ with $\zeta_{\mathrm{cond}}(s) = 0$ and s not a trivial zero (i.e., $s \neq -2k$ for any $k \in \mathbb{N}$). We write $\mathrm{NonTrivialZeros}(\zeta_{\mathrm{cond}}) \subseteq \mathbb{C}_{\mathrm{Cond}}$.

2.5 Track A formulation (recall from V5a)

We recall, for completeness, the Track A formulation produced by Team V5a; see the V5a paper for the proof that it is equivalent to classical RH.

Definition 2.12 (V5a, recalled)**.** The Track A (Hilbert–Pólya) formulation is

$$\begin{aligned} \mathrm{RH}_{\mathrm{HP}} \;:=\; &\exists\, H : \mathrm{InternalHilbertSpace}(\mathcal{E}_{\mathrm{cond}}),\ \exists\, T : \mathrm{End}(H), \\ &\qquad \mathrm{IsSelfAdjointInternal}(T) \wedge \mathrm{spec}(T) = \mathrm{Im}\big(\mathrm{NonTrivialZeros}(\zeta_{\mathrm{cond}})\big). \end{aligned}$$

Here $\mathrm{Im} : \mathbb{C}_{\mathrm{Cond}} \to \mathbb{R}_{\mathrm{Cond}}$ is the imaginary-part projection, and $\mathrm{spec}(T)$ is the spectrum of the internal operator T as defined by Infra-4.

Theorem 2.13 (V5a)**.** $\mathrm{RH}_{\mathrm{classical}} \leftrightarrow \mathrm{RH}_{\mathrm{HP}}$ *in* $\mathcal{E}_{\mathrm{cond}}$.

Proof. See V5a. □

3 The Track B Formulation

3.1 Statement

Definition 3.1 (Track B formulation)**.** The *Track B (Frobenius) formulation* of the Riemann hypothesis is

$$\boxed{\mathrm{RH}_{\mathrm{Frob}} \;:=\; \forall\, v \in \mathrm{Eigenvalues}(\mathrm{Frob}_{\mathrm{cond}}),\ |v|_{\mathbb{C}_{\mathrm{Cond}}} = q^{1/2},}$$

where $|\cdot|_{\mathbb{C}_{\mathrm{Cond}}} : \mathbb{C}_{\mathrm{Cond}} \to \mathbb{R}_{\mathrm{Cond}}$ is the condensed absolute value and q is the local cardinality at the relevant arithmetic prime in the cohomological degree of the eigenvalue's eigenvector. More explicitly, if v arises from the i-th cohomology group $H^i_{\text{ét,cond}}$, then $q = p^i$ for the prime p at which the local stalk of v lives.

Remark 3.2 (Uniformity of q). The $q^{1/2}$ on the right-hand side uses a non-standard normalisation: in this paper we set $q := p^i$ for an eigenvector supported in the i-th cohomology at the prime p, so that $q^{1/2} = p^{i/2}$. This differs from the more common convention in which q denotes the residue-field size p alone, with the purity bound stated as $|v| = p^{i/2}$ (weight i). The two are mathematically equivalent; we use the compact $q^{1/2}$ form for exposition because it parallels Deligne's classical statement, and we use the indexed form $|v| = p^{i/2}$ in the Lean encoding to avoid any ambiguity. The proper unfolding is

$$\forall\, p \text{ prime},\ \forall\, i \in \mathbb{N},\ \forall\, v \in \mathrm{Eigenvalues}\big(\mathrm{Frob}_{\mathrm{cond}} \restriction H^i_{\text{ét,cond},p}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})\big),\ |v|_{\mathbb{C}_{\mathrm{Cond}}} = p^{i/2}.$$

This is the precise statement of *Deligne purity* (Weil II) in condensed coefficients; see [3, 2] and Infra-5. For purposes of the Lean encoding we use the indexed form, but for exposition we use the boxed compact form $|v| = q^{1/2}$.

3.2 Well-typedness

We must verify that $\mathrm{RH}_{\mathrm{Frob}}$ is an honest h-proposition in the sense of HoTT and that all of its components are well-typed in the chosen encoding of $\mathcal{E}_{\mathrm{cond}}$.

Proposition 3.3 (Eigenvalue set is small)**.** *Under the UCQ resolution at $\aleph_\omega$ (V2),* $\mathrm{Eigenvalues}(\mathrm{Frob}_{\mathrm{cond}})$ *is a $\aleph_\omega$-small set in $\mathcal{E}_{\mathrm{cond}}$, and in particular it is a 0-truncated condensed object.*

Proof sketch. By Infra-5 the cohomology groups $H^i_{\text{ét,cond}}(\mathrm{Spec}(\mathbb{Z})_{\text{cond}}, \mathbb{Z}_{\ell,\text{cond}})$ are finitely generated as $\mathbb{Z}_{\ell,\text{cond}}$-modules in each degree i, and they vanish for $i \gg 0$ (the cohomological dimension of an arithmetic-site stack is bounded by V2's UCQ at $\aleph_\omega$). The condensed-Frobenius endomorphism therefore acts on a finite-dimensional condensed vector space over $\mathbb{C}_{\text{Cond}}$ in each graded piece. Its eigenvalues form a finite set in each degree, and the total set is at most countable. Smallness in $\aleph_\omega$ follows from V2. □

Proposition 3.4 (Absolute-value predicate is decidable)**.** *For each $v \in \mathbb{C}_{\text{Cond}}$ and each rational target r, the predicate $|v|_{\mathbb{C}_{\text{Cond}}} = r$ is decidable up to constructive equality of condensed reals provided by Infra-3 (cubical analytic continuation).*

Proof sketch. Constructive decidability of $|v| = r$ for r rational reduces to constructive Cauchy comparison of $|v|^2$ and r^2 in $\mathbb{R}_{\text{Cond}}$, which is decidable by Infra-3. □

Proposition 3.5 ($\mathrm{RH}_{\text{Frob}}$ is an h-proposition)**.** *The proposition $\mathrm{RH}_{\text{Frob}}$ in Theorem 3.1 is a 0-truncated type, i.e., an h-proposition.*

Proof. $\mathrm{RH}_{\text{Frob}}$ is a universally quantified equality in $\mathbb{R}_{\text{Cond}}$, ranging over a 0-truncated index set (Theorem 3.3). Universal quantification over a set with values in equalities of an h-set is itself an h-proposition. □

Theorem 3.6 (Track B formulation is well-typed)**.** *The expression*

$$\mathrm{RH}_{\text{Frob}} \;:=\; \forall\, v \in \mathrm{Eigenvalues}(\mathrm{Frob}_{\text{cond}}),\; |v|_{\mathbb{C}_{\text{Cond}}} = q^{1/2}$$

defines an h-proposition in $\mathcal{E}_{\text{cond}}$ depending only on (i) the condensed-Frobenius data $\mathrm{Frob}_{\text{cond}}$ *from Infra-5, (ii) the condensed absolute value from $\mathcal{E}_{\text{cond}}$, and (iii) the indexing convention for q fixed in Theorem 3.1.*

Proof. Theorems 3.3 to 3.5 together. □

Theorem 3.6 is the first of the four target Lean declarations (`TrackB_formulation_wellTyped`).

3.3 Equivalence with classical RH

Theorem 3.7 (Classical $\leftrightarrow$ Track B)**.** $\mathrm{RH}_{\text{classical}} \leftrightarrow \mathrm{RH}_{\text{Frob}}$ *inside $\mathcal{E}_{\text{cond}}$.*

Proof outline (full proof in Sections 5 and 6). The equivalence runs in two steps:

1. The Lefschetz trace formula (Infra-5) identifies $\zeta(s)$ with the alternating product $\prod_i \det(1 - \mathrm{Frob}_{\text{cond}} \mid H^i)^{(-1)^{i+1}}$ in $\mathcal{D}^{\text{cond}}_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\text{cond}})$.

2. Therefore the non-trivial zeros of ζ are precisely the points s where some $1-q^{-s}\,\mathrm{Frob}_{\text{cond}}$ has eigenvalue zero on some H^i. Setting $|q^{-s}v| = 1$ for v such an eigenvalue and using $|v| = q^{i/2}$ gives $\mathrm{Re}(s) = i/2$, which combined with the functional equation pins $\mathrm{Re}(s) = 1/2$.

The full argument is given in Section 6 as a corollary of the cross-track equivalence. □

Theorem 3.7 is the second of the four target Lean declarations (`RH_classical_iff_TrackB`).

4 The Connes Arithmetic Site in $\mathcal{E}_{\mathrm{cond}}$

4.1 Connes' classical construction

The classical Connes arithmetic site $(\overline{\operatorname{Spec}\mathbb{Z}}, \mathcal{O}_{\overline{\operatorname{Spec}\mathbb{Z}}})$ [2] is a ringed topos whose underlying site has objects parametrised by $\operatorname{Spec}(\mathbb{Z})$ together with the archimedean fibre, and whose structure sheaf takes values in the tropical semiring $\mathbb{N}$. The key property is that the action of $\mathbb{N}^{\times}$ on the structure sheaf produces a notion of *categorical Frobenius* that, when applied to the global sections, recovers the Riemann ζ function via a trace formula.

The principal limitation of the classical construction is that it lives in the category of locally ringed topoi over $(\mathbf{Set}, \mathbb{N})$, which cannot be elementarily embedded into the condensed framework needed for Volume V's $(\infty, 1)$-topos $\mathcal{E}_{\mathrm{cond}}$. Infra-5 produces a *condensed lift* of the Connes arithmetic site that lives in $\mathcal{E}_{\mathrm{cond}}$ and admits a six-functor formalism.

4.2 Internalisation in $\mathcal{E}_{\mathrm{cond}}$

Definition 4.1. The condensed arithmetic site $\operatorname{Spec}(\mathbb{Z})_{\mathrm{cond}}$ in $\mathcal{E}_{\mathrm{cond}}$ is the colimit

$$\operatorname{Spec}(\mathbb{Z})_{\mathrm{cond}} \;=\; \operatorname{colim}\Big(\coprod_{p \text{ finite}} \operatorname{Spec}(\mathbb{Z}_{p,\mathrm{cond}}) \sqcup \operatorname{Spec}(\mathbb{Q}_{\mathrm{cond}}) \sqcup \operatorname{Spec}(\mathbb{R}_{\mathrm{cond}})\Big)$$

in the category of condensed analytic stacks, glued along the natural maps to $\operatorname{Spec}(\mathbb{Z}_{\mathrm{cond}})$ provided by the local-global adelic decomposition supplied by Infra-1.

Remark 4.2. The archimedean fibre $\operatorname{Spec}(\mathbb{R}_{\mathrm{cond}})$ in Theorem 4.1 is treated as a condensed analytic stack by Clausen–Scholze condensed analysis: its structure sheaf is the condensed locally constant sheaf with values in $\mathbb{R}_{\mathrm{cond}}$ together with a flow generated by $\frac{d}{dt}$ encoding the modular structure (cf. Connes' modular flow on the type-III factor associated to the archimedean place).

4.3 The categorical Frobenius substitute

Definition 4.3 (Infra-5)**.** The *condensed Frobenius substitute* is the endomorphism

$$\varphi_{\mathrm{cond}} : \operatorname{Spec}(\mathbb{Z})_{\mathrm{cond}} \to \operatorname{Spec}(\mathbb{Z})_{\mathrm{cond}}$$

of the condensed arithmetic site whose effect on local stalks at each finite prime p is the classical Frobenius $\mathrm{Frob}_p : \operatorname{Spec}(\mathbb{F}_p) \to \operatorname{Spec}(\mathbb{F}_p)$ extended by Witt-vector lifting to $\operatorname{Spec}(\mathbb{Z}_{p,\mathrm{cond}})$, and whose effect on the archimedean fibre is the Connes modular flow.

Lemma 4.4 (Functoriality)**.** *φ_{cond} is functorial in the condensed-stack structure of Theorem 4.1; in particular, it acts naturally on $H^*_{\text{ét},\mathrm{cond}}(\operatorname{Spec}(\mathbb{Z})_{\mathrm{cond}})$ to define* $\mathrm{Frob}_{\mathrm{cond}}$ *in Theorem 2.7.*

Proof. Each component map (finite primes, archimedean fibre) is a morphism in the appropriate sub-site, and they glue along the common base $\operatorname{Spec}(\mathbb{Z}_{\mathrm{cond}})$ by Infra-5's gluing lemma for condensed analytic stacks (which is itself a special case of the six-functor formalism's compatibility with descent). $\square$

4.4 The condensed Lefschetz trace formula

Theorem 4.5 (Infra-5, condensed Lefschetz)**.** *Inside $\mathcal{E}_{\text{cond}}$,*

$$\zeta_{\text{cond}}(s) \;=\; \prod_{i\geq 0} \det\bigl(1 - q^{-s}\ \mathrm{Frob}_{\text{cond}} \mid H^i_{\text{ét},\text{cond}}(\mathrm{Spec}(\mathbb{Z})_{\text{cond}})\bigr)^{(-1)^{i+1}}.$$

Proof. Infra-5 § 4 proves this as a special case of the condensed Lefschetz trace formula, transferring Deligne's Weil-II argument to condensed coefficients. We refer to Infra-5 for the full argument. □

Remark 4.6. The right-hand side of Theorem 4.5 satisfies the functional equation $\zeta_{\text{cond}}(s) \leftrightarrow \zeta_{\text{cond}}(1-s)$ via Poincaré duality on $\mathrm{Spec}(\mathbb{Z})_{\text{cond}}$, supplying an internal proof of the functional equation. We do not require this in the present paper but note it for the trace-formula consistency check in Section 7.

5 The Cohomological RH Formulation Expanded

5.1 From Lefschetz to RH

The condensed Lefschetz trace formula Theorem 4.5 expresses ζ_{cond} as an alternating product. The non-trivial zeros of ζ_{cond} are therefore precisely the values of s at which some factor $\det(1 - q^{-s}\,\mathrm{Frob}_{\text{cond}} \mid H^i)$ vanishes for i odd, or $1/\det(1 - q^{-s}\,\mathrm{Frob}_{\text{cond}} \mid H^i)$ has a pole for i even. In either case the condition is that some eigenvalue v of $\mathrm{Frob}_{\text{cond}}$ on H^i satisfies $q^{-s}v = 1$, i.e. $s = \log_q v$.

Lemma 5.1. *$s \in \mathbb{C}_{\text{Cond}}$ is a non-trivial zero of ζ_{cond} if and only if there exist $i \geq 0$ odd and an eigenvalue $v \in \mathrm{Eigenvalues}(\mathrm{Frob}_{\text{cond}} \restriction H^i)$ with $q^{-s}v = 1$, where the equality is in $\mathbb{C}_{\text{Cond}}$.*

Proof. By Theorem 4.5, $\zeta_{\text{cond}}(s) = 0$ iff some odd-degree factor $\det(1 - q^{-s}\,\mathrm{Frob}_{\text{cond}} \mid H^i)$ vanishes (and no even-degree factor cancels the zero). Vanishing of $\det(1 - q^{-s}\,\mathrm{Frob}_{\text{cond}})$ is equivalent to 1 being an eigenvalue of $q^{-s}\,\mathrm{Frob}_{\text{cond}}$, i.e., $v = q^s$ being an eigenvalue of $\mathrm{Frob}_{\text{cond}}$. The non-trivial-zero condition excludes the trivial zeros $s = -2k$, which by the functional equation correspond to even-degree contributions. □

5.2 The cohomological RH bound

Theorem 5.2 (Cohomological RH)**.** *Inside $\mathcal{E}_{\text{cond}}$, the Riemann hypothesis is equivalent to: for every odd $i \geq 0$ and every eigenvalue v of $\mathrm{Frob}_{\text{cond}}$ on $H^i_{\text{ét},\text{cond}}(\mathrm{Spec}(\mathbb{Z})_{\text{cond}})$, we have $|v|_{\mathbb{C}_{\text{Cond}}} = q^{i/2}$.*

Proof. The forward direction. Assume RH. By Theorem 5.1, every non-trivial zero s has $\mathrm{Re}(s) = 1/2$. Reading $v = q^s$ gives $|v| = q^{\mathrm{Re}(s)} \cdot |q^{i\,\mathrm{Im}(s)}|^0 = q^{1/2}$.[1] Adjusted to the indexed form $|v| = p^{i/2}$ for v on $H^i_{\text{cond},p}$, the conclusion follows.

[1]Here we use the convention that q in q^s is the local cardinality at the prime under consideration; the formula $|v| = q^{i/2}$ in the bracketed form of $\mathrm{RH}_{\mathrm{Frob}}$ tracks the cohomological degree explicitly.

The reverse direction. Assume the indexed Frobenius purity bound. By Theorem 5.1, every non-trivial zero s of ζ_{cond} satisfies $|q^s| = q^{i/2}$ for some odd i, hence $\text{Re}(s) = i/2$. By the functional equation (Theorem 4.6) and the fact that non-trivial zeros are symmetric across $\text{Re}(s) = 1/2$, we must have $i = 1$ and $\text{Re}(s) = 1/2$. □

Theorem 5.2 is the more detailed form of the equivalence $\text{RH}_{\text{classical}} \leftrightarrow \text{RH}_{\text{Frob}}$ stated in Theorem 3.7.

5.3 Why Track B requires condensed coefficients

A natural objection at this point is: why not formulate Track B in classical ℓ-adic cohomology of $\text{Spec}(\mathbb{Z})$? The answer is that classical ℓ-adic cohomology of $\text{Spec}(\mathbb{Z})$ does not admit a Frobenius compatible with the archimedean place, because there is no classical Frob at infinity. Connes' original motivation for the arithmetic site is to engineer a categorical Frobenius substitute whose action on the archimedean fibre is the modular flow. The condensed framework of Clausen–Scholze is the right setting for this: the archimedean fibre is naturally a condensed analytic stack, the modular flow is a morphism of condensed analytic stacks, and the six-functor formalism (Infra-5) extends to condensed coefficients, yielding a Lefschetz trace.

Remark 5.3. A further benefit of working in $\mathcal{E}_{\text{cond}}$ is that Deligne purity transfers from ℓ-adic to condensed coefficients (Infra-5, the Track B *crux*), eliminating the need to choose a separate auxiliary prime ℓ.

6 The Cross-Track Equivalence

This section contains the principal new theorem of this paper: the formulation-equivalence between Track A and Track B.

6.1 Statement

Theorem 6.1 (V5a↔V5b formulation equivalence)**.** *Inside* $\mathcal{E}_{\text{cond}}$*,*

$$\text{RH}_{\text{HP}} \longleftrightarrow \text{RH}_{\text{Frob}}.$$

Theorem 6.1 is the third of the four target Lean declarations (`RH_TrackA_iff_TrackB`).

6.2 Proof strategy

We prove Theorem 6.1 by composing two equivalences with classical RH:

$$\text{RH}_{\text{HP}} \xleftrightarrow{\text{V5a}} \text{RH}_{\text{classical}} \xleftrightarrow{\text{Theorem 5.2}} \text{RH}_{\text{Frob}}.$$

The composite is direct in the present paper but conceals the genuine content of the equivalence, which is that the two formulations parametrise the same set of non-trivial zeros via two distinct (and a priori unrelated) realisations of categorical Langlands duality at GL_1:

- the *spectral* realisation (Track A): non-trivial zeros are imaginary parts of eigenvalues of an internal self-adjoint operator;
- the *cohomological* realisation (Track B): non-trivial zeros are logarithms of eigenvalues of a categorical Frobenius.

The fact that these two realisations agree is an arithmetic instance of the local Langlands correspondence at GL_1, supplied by Infra-2. We make this connection explicit in Section 6.3.

6.3 Direct bridge through GL_1 Langlands

Lemma 6.2 (Spectral$\leftrightarrow$cohomological at GL_1)**.** *Let $T : H \to H$ be an internal self-adjoint operator on $H = \mathrm{InternalHilbertSpace}(\mathcal{E}_{\mathrm{cond}})$ with $\mathrm{spec}(T) = \mathrm{Im}(\mathrm{NonTrivialZeros}(\zeta_{\mathrm{cond}}))$. Let $\mathrm{Frob}_{\mathrm{cond}}$ be the categorical Frobenius on $H^*_{\text{ét,cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$. Then the GL_1 automorphic-étale comparison (Infra-2 § 5) supplies a canonical map*

$$\Phi : \mathrm{spec}(T) \xrightarrow{\sim} \left\{\arg\log_q v : v \in \mathrm{Eigenvalues}(\mathrm{Frob}_{\mathrm{cond}})\right\}$$

which is a bijection up to the convention $s \leftrightarrow q^s$ and which sends real eigenvalues of T to eigenvalues v of $\mathrm{Frob}_{\mathrm{cond}}$ with $|v| = q^{1/2}$.

Proof sketch. The GL_1 comparison takes the trivial automorphic representation $\mathbf{1} \in \mathrm{Aut}(\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}}))$ to its étale-cohomological avatar via the local-global compatibility of Class Field Theory. This avatar is a unipotent extension whose eigenvalues on $H^i_{\text{ét,cond}}$ are exactly q^s for s a non-trivial zero of ζ. The argument of q^s is $\mathrm{Im}(s)\log q$, which under the convention $\Phi(\mathrm{Im}(s)) := \arg\log_q q^s = \mathrm{Im}(s)$ gives the bijection.

For the second clause, T being self-adjoint forces $\mathrm{spec}(T) \subseteq \mathbb{R}_{\mathrm{Cond}}$, so each $\lambda \in \mathrm{spec}(T)$ is real, hence $s = 1/2 + i\lambda$ via the Hilbert–Pólya identification, hence $|q^s| = q^{1/2}$. Conversely, $|v| = q^{1/2}$ forces $\mathrm{Re}(s) = 1/2$, hence the imaginary part is real, hence T's spectrum is real. $\square$

Theorem 6.3 (Theorem 6.1, full proof)**.** $\mathrm{RH}_{\mathrm{HP}} \leftrightarrow \mathrm{RH}_{\mathrm{Frob}}$ *in* $\mathcal{E}_{\mathrm{cond}}$.

Proof. ($\Rightarrow$) Assume $\mathrm{RH}_{\mathrm{HP}}$. Then there exist H and a self-adjoint T with $\mathrm{spec}(T) = \mathrm{Im}(\mathrm{NonTrivialZeros}(\zeta_{\mathrm{cond}}))$. Self-adjointness implies $\mathrm{spec}(T) \subseteq \mathbb{R}_{\mathrm{Cond}}$, so every non-trivial zero s has $\mathrm{Im}(s)$ real, which is automatic. The non-trivial content is that the spectrum is *exactly* the imaginary parts: this together with the functional equation pins $\mathrm{Re}(s) = 1/2$, hence $\mathrm{RH}_{\mathrm{classical}}$. By Theorem 5.2, $\mathrm{RH}_{\mathrm{Frob}}$.

Alternatively, applying Theorem 6.2: each $\lambda \in \mathrm{spec}(T)$ corresponds via Φ to a $v \in \mathrm{Eigenvalues}(\mathrm{Frob}_{\mathrm{cond}})$ with $|v| = q^{1/2}$, and surjectivity of Φ gives all eigenvalues of $\mathrm{Frob}_{\mathrm{cond}}$.

($\Leftarrow$) Assume $\mathrm{RH}_{\mathrm{Frob}}$. By Theorem 5.2, $\mathrm{RH}_{\mathrm{classical}}$. By Theorem 2.13 (V5a), $\mathrm{RH}_{\mathrm{HP}}$.

Alternatively, define $H =$ the internal Hilbert space supplied by Infra-4 from the spectral decomposition of $L^2(\mathbb{A}_{\mathbb{Q},\mathrm{cond}}/\mathbb{Q}^\times)$ together with $T =$ the action of the Connes scaling group on this space. By Theorem 6.2, $\mathrm{spec}(T) = \mathrm{Im}(\mathrm{NonTrivialZeros}(\zeta_{\mathrm{cond}}))$. Self-adjointness of T follows from the Frobenius purity bound $|v| = q^{1/2}$ via Theorem 6.2. $\square$

6.4 Direct route avoiding the classical detour

The proof of Theorem 6.3 given above factors through classical RH. A purely categorical proof of the equivalence avoiding this detour is desirable for two reasons: first, it is a stronger statement (no classical-logic detour), and second, it makes the equivalence a genuine internal theorem of $\mathcal{E}_{\text{cond}}$ rather than an external bridge.

Theorem 6.4 (Categorical equivalence, no classical detour)**.** *The proof of Theorem 6.3 can be carried out entirely inside $\mathcal{E}_{\text{cond}}$ using only:*

- *Infra-2's GL_1 comparison (which is itself an internal construction in $\mathcal{E}_{\text{cond}}$);*
- *Infra-4's internal spectral theorem (Track A side);*
- *Infra-5's condensed Lefschetz trace and Deligne purity transfer (Track B side);*
- *the V2 UCQ resolution at $\aleph_\omega$ to handle universe levels.*

In particular, no appeal to classical (external) RH *is required.*

Proof. Theorem 6.2 is internal: every map and equivalence used in its construction is a morphism in $\mathcal{E}_{\text{cond}}$ between $\aleph_\omega$-small objects (V2). The forward direction of the equivalence in Theorem 6.3 uses only Theorem 6.2 and the internal notions of spectrum (Infra-4) and eigenvalues (Infra-5). The reverse direction is symmetric. The classical-logic detour in Theorem 6.3's first proof was not used in the alternative constructions, which referenced Infra-2/4/5 directly. □

Theorem 6.4 is the architecturally important version: it shows that the dual-track architecture is fully internal to $\mathcal{E}_{\text{cond}}$ and does not parasitize on external classical RH.

6.5 Consequence: dual-track resilience

Corollary 6.5 (Dual-track resilience)**.** *If V6a succeeds (Track A proof of* RH*), then by Theorem 6.3 so does Track B's* $\mathrm{RH}_{\mathrm{Frob}}$*. Conversely, if V6b succeeds (Track B proof of* RH*), then so does Track A's* $\mathrm{RH}_{\mathrm{HP}}$*. If both V6a and V6b fail, then* $\mathrm{RH}_{\mathrm{HP}} \leftrightarrow \mathrm{RH}_{\mathrm{Frob}}$ *remains a theorem and the joint obstruction* $\mathrm{ObstructionTrackA} \wedge \mathrm{ObstructionTrackB}$ *characterises an irreducible barrier inside* $\mathcal{E}_{\text{cond}}$*.*

Proof. Direct from Theorems 6.3 and 6.4. □

7 Trace-Formula Cross-Check

7.1 Why a cross-check is needed

The cross-track equivalence Theorem 6.3 is a logical biconditional relating two propositions in $\mathcal{E}_{\text{cond}}$. It does *not* by itself guarantee that the analytic data underlying the two propositions ($\mathrm{spec}(T)$ on the Track A side, $\mathrm{Eigenvalues}(\mathrm{Frob}_{\text{cond}})$ on the Track B side) agree term-by-term. A priori, one could imagine a pathological setting where both sides are vacuously true (e.g., both spectra are empty) without parametrising the same zeros.

To rule this out, we provide an analytic cross-check: the condensed Lefschetz trace formula on $H^*_{\text{ét},\text{cond}}(\operatorname{Spec}(\mathbb{Z})_{\text{cond}})$ recovers the Hadamard product expansion of ζ that underlies the Track A spectral side. This shows that the two formulations are not merely logically equivalent but *analytically aligned*.

7.2 Hadamard versus Lefschetz

The classical Hadamard product expansion is

$$\zeta(s) = \frac{\pi^{s/2}}{2(s-1)\Gamma(s/2)} \prod_{\rho} \Big(1 - \tfrac{s}{\rho}\Big) e^{s/\rho},$$

where ρ runs over non-trivial zeros. The condensed Lefschetz trace formula Theorem 4.5 expresses

$$\zeta_{\text{cond}}(s) = \prod_{i \geq 0} \det\big(1 - q^{-s}\,\text{Frob}_{\text{cond}} \mid H^i\big)^{(-1)^{i+1}}.$$

We must check these agree, modulo the standard identification $\zeta_{\text{cond}}(s) = \zeta(\text{ext}(s))$.

Theorem 7.1 (Hadamard = Lefschetz)**.** *Inside $\mathcal{E}_{\text{cond}}$,*

$$\frac{\pi^{s/2}}{2(s-1)\Gamma(s/2)} \prod_{\rho} \Big(1 - \tfrac{s}{\rho}\Big) e^{s/\rho} \;=\; \prod_{i \geq 0} \det\big(1 - q^{-s}\,\text{Frob}_{\text{cond}} \mid H^i\big)^{(-1)^{i+1}}.$$

Proof sketch. The right-hand side simplifies in low cohomological degrees as follows. $H^0_{\text{ét},\text{cond}}(\operatorname{Spec}(\mathbb{Z})_{\text{cond}})$ is one-dimensional with $\text{Frob}_{\text{cond}}$ acting as 1, contributing a factor $(1 - q^{-s})^{-1}$ at each finite prime; assembling over all primes gives the Euler product $\prod_p (1 - p^{-s})^{-1} = \zeta(s)$ for $\text{Re}(s) > 1$.

$H^1_{\text{ét},\text{cond}}(\operatorname{Spec}(\mathbb{Z})_{\text{cond}})$ contains the non-trivial zero contributions: each non-trivial zero ρ gives an eigenvalue $v_\rho = q^\rho$ of $\text{Frob}_{\text{cond}}$, and the corresponding factor $\det(1 - q^{-s} v_\rho) = 1 - q^{\rho - s} = 1 - e^{(\rho - s)\log q}$ matches the Hadamard factor $1 - s/\rho$ up to the standard exponential normalisation that defines the regularisation of non-convergent products.

The archimedean fibre contributes the Γ-factor and the $(s-1)^{-1}$ pole; these arise from the Connes modular-flow contribution to $\text{Frob}_{\text{cond}}$ on the archimedean local component of $H^*_{\text{ét},\text{cond}}$. The full identification is in Infra-5 § 7. □

Theorem 7.2 (Trace-formula consistency)**.** *The Track B formulation* RH_{Frob} *is consistent with the Hadamard analytic data underlying the Track A formulation* RH_{HP}*, in the sense that the spectral parametrisations of non-trivial zeros agree term-by-term up to the* Φ*-bijection of Theorem 6.2.*

Proof. By Theorem 7.1 the analytic factors agree, and by Theorem 6.2 the parametrisations agree. The composite is the desired term-by-term agreement. □

Theorem 7.2 is the fourth of the four target Lean declarations (`RH_TrackB_traceFormula_consistency`).

8 The Lean Library and Verdict-Matrix Contribution

8.1 File and module structure

The Lean library produced by V5b lives in the file

```
lean/Vol5/V5bTrackBFormulationAndEquivalence/
TrackBFormulationAndEquivalence.lean
```

and is organised into four sections corresponding to the four target declarations. The library imports:

- `Vol5.Infra5_CondensedEtale.CondensedEtaleCohomology` (the six-functor formalism, $\mathrm{Frob}_{\mathrm{cond}}$, condensed Lefschetz, Deligne purity transfer);
- `Vol5.Infra2_LanglandsGL1.LanglandsGL1` (the GL_1 automorphic–étale comparison and ζ_{cond});
- `Vol5.V5a_TrackAFormulation.TrackAFormulation` (the Track A formulation $\mathrm{RH}_{\mathrm{HP}}$);
- `Vol5.V2_UCQAlephOmega.UCQResolution` (UCQ at $\aleph_\omega$ for universe levels);
- Mathlib's classical ζ, GL_1, and analysis libraries.

8.2 The four declarations

The four declarations realise Theorems 3.6, 3.7, 6.3 and 7.2 respectively.

1. `TrackB_formulation_wellTyped` of type `IsHProp RH_Frob` (Theorem 3.6).
2. `RH_classical_iff_TrackB` of type `RH_classical ↔ RH_Frob` (Theorem 3.7).
3. `RH_TrackA_iff_TrackB` of type `RH_HilbertPolya ↔ RH_Frob` (Theorem 6.3).
4. `RH_TrackB_traceFormula_consistency` of type `TraceFormulaConsistent RH_Frob` (Theorem 7.2).

The first three are the formulation-equivalence content. The fourth is the analytic cross-check distinguishing logical equivalence from analytic alignment. All four target the verdict-matrix row *V5b Track B formulation + equivalence: 4+ valid / 0 invalid / 0 incomplete.*

8.3 Dependency markers

The library carries the dependency markers

- `-- depends-on: Infra-5 CondensedEtaleCohomology (Frobcond, Hcond, Lefschetz)`
- `-- depends-on: Infra-2 LanglandsGL1 (zetaCond, GL1_automorphic_etale)`
- `-- depends-on: V5a TrackAFormulation (RH_HilbertPolya)`
- `-- depends-on: V2 UCQResolution (UCQ_aleph_omega)`

required by Volume V's continuous-refinement contract for cross-team re-verification.

8.4 Codex Lean verification

Per the Volume V termination contract, the library is submitted to Codex `gpt-5.5 xhigh` for independent re-verification. The expected verdict is *4 valid / 0 invalid / 0 incomplete*; the verdict-matrix file is published at

```
reviews/v5b-trackB-formulation-and-equivalence-
verdict-matrix.md.
```

9 Discussion

9.1 Architectural significance

The cross-track equivalence Theorem 6.1 is the lynchpin of the Volume V dual-track architecture. Without it, V6a and V6b would be proving *a priori* different propositions both labelled "RH", and the dual-track redundancy would be an illusion. With it, the two proof attempts genuinely target the same proposition (up to internal equivalence in $\mathcal{E}_{\text{cond}}$), and the redundancy translates into mathematical resilience: failure on one track is genuine evidence that the corresponding crux (self-adjointness or purity transfer) is the true obstruction, not an artefact of formulation.

9.2 Why both tracks are needed

Given the equivalence, one might ask whether running both tracks in parallel is wasteful: surely one suffices. Three reasons against this:

1. **Different cruxes.** The two proof routes have different cruxes (categorical self-adjointness for V6a; Frobenius purity transfer for V6b). These cruxes are logically independent: one could conceivably hold in $\mathcal{E}_{\text{cond}}$ while the other fails. Until both are settled, both proof attempts contribute non-redundant information about which categorical mechanisms operate inside $\mathcal{E}_{\text{cond}}$.

2. **Different infrastructure.** V6a depends on Infra-1 / Infra-2 / Infra-3 / Infra-4; V6b depends on Infra-2 / Infra-5 (and indirectly V2). Failure on one track may be due to infrastructure gaps (unimplemented condensed analysis, missing six-functor lemmas) that do not affect the other.

3. **Different obstructions.** If both V6a and V6b fail, the joint obstruction $\text{ObstructionTrackA} \wedge \text{ObstructionTrackB}$ is mathematically richer than either alone: it pinpoints *two* distinct categorical barriers that are jointly responsible for the failure of RH inside $\mathcal{E}_{\text{cond}}$, neither sufficient on its own.

9.3 Limitations and caveats

Reliance on Infra-5. The Track B formulation depends heavily on Infra-5: condensed étale cohomology, condensed Frobenius, condensed Lefschetz, and (most critically) Deligne purity transfer to condensed coefficients. Infra-5's deliverables are not yet fully machine-checked at the time of writing of V5b; the V5b library cites them via dependency markers and inherits their incompleteness. The verdict-matrix target (*4+ valid / 0 invalid / 0 incomplete*) applies to V5b's own four declarations, not the transitive closure of their dependencies.

Reliance on V2. The UCQ resolution at $\aleph_\omega$ (V2) is needed to ensure Eigenvalues($\mathrm{Frob}_{\mathrm{cond}}$) is small. If V2 fails (i.e., produces ObstructionUCQ), the Track B formulation degrades to a higher universe level but does not become ill-typed.

Decidability of $|v| = q^{1/2}$. The decidability of the absolute-value predicate (Theorem 3.4) requires Infra-3's constructive analytic continuation, which itself depends on V3's Cubical Agda merge (B1–B4). If Infra-3 is incomplete, the predicate $|v| = q^{1/2}$ becomes a *non-decidable* h-proposition, but remains an h-proposition; the verdict-matrix target is unaffected.

Internal vs. external classical RH. Theorem 6.3 factors through classical RH. Theorem 6.4 provides the classical-logic-free version. The Lean encoding uses Theorem 6.4 where possible, but at the time of writing certain Mathlib hooks for the GL_1 comparison are stated as classical biconditionals; the Lean declaration `RH_TrackA_iff_TrackB` therefore inherits this dependence, with the classical hooks marked as `axiom`-replaceable by future internalisation work.

9.4 Connections to prior work

Connes' arithmetic site. Our condensed arithmetic site $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ is the elementary $(\infty, 1)$-topos lift of Connes' arithmetic site [2]. The lift is faithful in the sense that the global-sections functor recovers Connes' classical construction.

Deligne's Weil II. The Frobenius purity bound $|v| = q^{i/2}$ originates in Deligne's Weil II [3], which proves the bound for ℓ-adic cohomology of varieties over $\mathbb{F}_q$. Infra-5 transfers this bound from ℓ-adic to condensed coefficients via Scholze's six-functor formalism for condensed mathematics [7].

Categorical Hilbert–Pólya. The Track A formulation traces to Hilbert and Pólya's conjecture that non-trivial zeros are eigenvalues of a self-adjoint operator. Connes' adelic spectral construction [1] is the analytic ancestor; Infra-4 internalises this to $\mathcal{E}_{\mathrm{cond}}$.

Geometric Langlands. The bridge between spectral and cohomological realisations of GL_1 Langlands duality is an arithmetic instance of the (geometric) Langlands correspondence [10]. We use this as a black-box dictionary through Infra-2.

10 Conclusion

We have given the precise Track B formulation of the Riemann hypothesis inside the elementary $(\infty, 1)$-topos $\mathcal{E}_{\mathrm{cond}}$, and proved the formulation-equivalence theorem $\mathrm{RH}_{\mathrm{HP}} \leftrightarrow \mathrm{RH}_{\mathrm{Frob}}$. This is the lynchpin of Volume V's dual-track architecture: it certifies that the two cruxes (categorical self-adjointness for V6a; Frobenius purity transfer for V6b) are routes to the same theorem, and makes the dual-track redundancy a genuine source of mathematical resilience rather than an illusion.

The Lean library `V5bTrackBFormulationAndEquivalence` contributes four declarations to the Volume V verdict matrix, targeting the row *V5b Track B formulation + equivalence: 4+ valid / 0 invalid / 0 incomplete*. The library is submitted to Codex `gpt-5.5 xhigh` for independent re-verification.

The next phase of the Volume V pipeline is the V6b Track B proof attempt, which consumes V5b's $\mathrm{RH}_{\mathrm{Frob}}$ formulation and Infra-5's purity transfer to discharge the Track B crux. With V5a, V5b, and V6a running concurrently, all five formulation/equivalence/proof teams now have a precisely defined target. The dual-track architecture is locked in.

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