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Track A Attempt 1
Adelic spectral construction → Selberg–Connes trace → categorical self-adjointness → functional equation.
Crux: Step 3 (self-adjointness).
Outcome: Obstruction.
`theorem ObstructionTrackA` captures the polarization gap exactly.

Toolchain: Lean 4 / Mathlib `0f9072d`.
Verifier: Codex `gpt-5.5 xhigh`.

Volume V, Team V6a — Track A Proof Attempt: Adelic Spectral Construction, Categorical Self-Adjointness, and the Riemann Hypothesis

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2026-05-06 (attempt-1)

Abstract

We pursue the Track A proof of the Riemann Hypothesis via the categorical Hilbert–Pólya programme inside the condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\text{cond}}$. The four-step structure is: (1) construct the Connes spectral decomposition of $L^2(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)$ as an internal Hilbert space $\mathcal{H}$ in $\mathcal{E}_{\text{cond}}$; (2) establish the categorical Selberg–Connes trace formula equating geometric and spectral sides; (3) prove the categorical trace operator $\mathcal{T}$ is internally self-adjoint, using the symplectic structure imported from analytic Langlands at GL_1 — *the Track A crux*; (4) conclude via the Infra-4 internal spectral theorem and the V5a Hilbert–Pólya bridge that $\mathrm{spec}(\mathcal{T}) \subseteq \mathbb{R}$, hence $\mathrm{Re}(s) = \frac{1}{2}$ for non-trivial zeros. We complete Steps 1, 2 and 4 modulo declared upstream axioms (`Infra-1`, `Infra-2`, `Infra-4`, `V5a`). Step 3 *does not close*. The obstruction is precise: the symplectic structure on $\mathcal{H}$ canonically supplies a complex pre-symplectic form, but extracting from it an *internally self-adjoint* Hamiltonian requires a $\mathcal{E}_{\text{cond}}$-internal compatible polarization, and we show that no such polarization is constructible from the data internal to $\mathcal{E}_{\text{cond}}$ alone — it would require an honest analytic uniformization that is exactly the Hilbert–Pólya content. We commit `theorem ObstructionTrackA` characterising the barrier as *the absence of a $\mathcal{E}_{\text{cond}}$-internal compatible polarization* on the Connes symplectic representation, and we prove a partial-positivity result reducing the full self-adjointness statement to a pre-existing classical input. We refuse to fake-close the proof. The companion Lean library `lean/Vol5/V6aTrackAProofAttempt/` formalises the four-step skeleton; the obstruction theorem is commit-

ted in `lean/Vol5/ObstructionTrackA.lean`; the proof attempt is committed as `lean/Vol5/RiemannHypothesis_TrackA_attempt-1.lean`.

Contents

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1 Introduction

1.1 Position within Volume V

Volume V of the HoTT–Riemann Hypothesis programme runs all 13 teams concurrently from day one. The proof teams V6a (Track A, Hilbert–Pólya) and V6b (Track B, Connes–Deligne) continuously re-attempt the proof against the current state of the eleven upstream libraries. This paper records the *first attempt* of Team V6a: `attempt-1`, dated 2026-05-06.

The four-step Track A structure is dictated by the Volume IV roadmap [18] (§6, OQ6) and the Volume V mission brief (§1.2):

Step 1. Adelic spectral construction (consumes Infra-1, Infra-2, Infra-4).

Step 2. Categorical Selberg–Connes trace formula (consumes Infra-3).

Step 3. Categorical self-adjointness (consumes Infra-4) — **the crux**.

Step 4. Conclusion via functional equation (consumes V5a).

If Step 3 closes, the headline theorem

$$\texttt{theorem RiemannHypothesis} : \forall s\colon \mathbb{C}_{\text{Cond}},\ s \in \text{NonTrivialZeros}(\zeta_{\text{cond}}) \Rightarrow s.\text{Re} = \tfrac{1}{2}$$

is committed. If it does not close, we instead commit a *precise* `theorem ObstructionTrackA` localising the irreducible barrier. The plan accepts both end-states, and the honesty contract forbids fake-closure with hidden axioms.

1.2 The honest verdict of attempt-1

Step 3 does not close. The categorical structure of $\mathcal{E}_{\text{cond}}$ gives us, after Steps 1–2, a complex pre-symplectic structure ω on the internal Hilbert space $\mathcal{H} = L^2_{\text{cond}}(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)$. To turn ω into a self-adjoint Hamiltonian one needs a *compatible polarization* $J\colon \mathcal{H} \to \mathcal{H}$ with $J^2 = -\mathbf{1}$, $\omega(\cdot, J\cdot)$ a positive inner product, and J commuting with the dilation flow. The categorical machinery of Infra-1 + Infra-2 produces ω, and Infra-4 packages the spectral theorem; but the polarization J is *not produced* by any of these. Producing J inside $\mathcal{E}_{\text{cond}}$ is precisely what the Hilbert–Pólya conjecture requires, and our analysis shows that it cannot

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be done from $\mathcal{E}_{\mathrm{cond}}$-internal data alone — it requires an external analytic input (an explicit Schauder basis for the dilation eigenfunctions) that is not constructively available in $\mathcal{E}_{\mathrm{cond}}$.

We present in §6 the formal `theorem ObstructionTrackA`, the partial-positivity reduction that converts the gap into a single quantitative classical question, and an analysis of which upstream advances would close it.

1.3 Outline

§2 reviews the categorical framework: $\mathcal{E}_{\mathrm{cond}}$, internal Hilbert spaces, internal self-adjointness, the Connes adelic data, and the upstream axioms cited. §3 works out Step 1 (adelic spectral construction). §4 works out Step 2 (Selberg–Connes trace formula). §5 is the crux: we set up the symplectic data, attempt the self-adjointness argument, and identify the polarization obstruction. §6 states and proves `theorem ObstructionTrackA`, including the partial-positivity reduction. §7 works out Step 4 conditional on Step 3, recording the would-be closure. §8 describes the Lean library, the axiom audit, and the verdict matrix. §9 compares with the classical Hilbert–Pólya literature, Connes' trace approach, and Berry–Keating physics. §10 discusses outlook for V6a re-attempts. §11 concludes.

1.4 Conventions

We work entirely inside $\mathcal{E}_{\mathrm{cond}}$, the condensed elementary $(\infty, 1)$-topos of Volume IV §OQ5; all categorical constructions are internal to $\mathcal{E}_{\mathrm{cond}}$ unless stated. "Internal Hilbert space" means a sheaf of Hilbert spaces equipped with the inner-product structure of Infra-4 §2. Classical statements (denoted *ext.*) refer to Mathlib's standard real analysis. The classical Riemann zeta function is $\zeta(s)$; its non-trivial zeros are

$$Z := \{s \in \mathbb{C} : 0 < \mathrm{Re}(s) < 1,\ \zeta(s) = 0\},$$

and the conjecture is $Z \subseteq \{\mathrm{Re}(s) = \frac{1}{2}\}$.

2 Mathematical Framework

2.1 The condensed elementary $(\infty, 1)$-topos

We adopt the Volume IV definition.

Definition 2.1 ([1])**.** The *condensed elementary $(\infty, 1)$-topos* $\mathcal{E}_{\mathrm{cond}}$ is the elementary $(\infty, 1)$-topos satisfying Rasekh's axioms (E1)–(E5) (see [15, 10]) whose underlying ∞-topos is the ∞-topos Cond_∞ of light condensed anima of Clausen–Scholze [11]. Its internal logic is the higher-order intuitionistic logic with Univalence, NNO, and the Rasekh closure properties; cf. the formal status of condensed mathematics in Lean as established by the Liquid Tensor Experiment [16].

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We invoke the following internal facts from Volume IV and from Infra-1, -2, -4 (cited as upstream axioms of the present paper):

- Internal complex numbers $\mathbb{C}_{\mathrm{Cond}}$, additive group, multiplicative monoid, conjugation.
- Internal Hilbert spaces $\mathcal{H}$: a triple of an internal $\mathbb{C}_{\mathrm{Cond}}$-vector space, an inner product $\langle -, - \rangle \colon \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}_{\mathrm{Cond}}$, and a completeness witness in the sense of Infra-4.
- For $T \colon \mathcal{H} \to \mathcal{H}$ internally bounded, the adjoint T^* exists and is unique; T is internally self-adjoint iff $T = T^*$.
- The Riemann zeta as the GL_1-special automorphic L-function $\zeta_{\mathrm{cond}} := L(s, \mathbf{1})$ (Infra-2).

2.2 Adeles, idele class group, and dilation flow

Definition 2.2. The *adele ring* of $\mathbb{Q}$ is the restricted product $\mathbb{A}_{\mathbb{Q}} := \mathbb{R} \times \prod'_{p\,\mathrm{prime}} \mathbb{Q}_p$, taken inside $\mathcal{E}_{\mathrm{cond}}$ via the condensed structure on each factor. The *idele class group* is $C_{\mathbb{Q}} := \mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$, equivalently $\mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^{\times}$ via the canonical splitting; we write $|\cdot| \colon C_{\mathbb{Q}} \to \mathbb{R}_{>0}$ for the idele norm.

Definition 2.3. The *dilation flow* is the action of $\mathbb{R}_{>0} \subset C_{\mathbb{Q}}$ on $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^{\times}$ by multiplication, with infinitesimal generator $D := -i\,t\,\partial_t$ in the archimedean coordinate.

2.3 The internal L^2-space

Construction 2.4. Define

$$\mathcal{H} := L^2_{\mathrm{cond}}(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^{\times})$$

as the internal Hilbert space in $\mathcal{E}_{\mathrm{cond}}$ obtained by sheafifying the classical L^2-construction with respect to the unique $C_{\mathbb{Q}}$-invariant Haar measure on $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^{\times}$. This relies on axiom `ML_LangCond1` (Infra-2 §3.4), a placeholder for a forthcoming theorem of the Infra-2 team, recorded with the marker `-- TODO: replaced by Infra-2` in the Lean library; by Infra-4 §2 the resulting object admits the spectral-theorem package.

Remark 2.5. The classical fact that $\mathcal{H}$ carries a unitary representation of $C_{\mathbb{Q}}$ via translation persists internally; see [2, 3]. We will need this in §3.

2.4 Cited upstream axioms

We are explicit about which results we cite from upstream teams, marking each as a temporary axiom. As Phase 3.5 lands the upstream commits, these axioms get replaced.

Axiom marker	Status
`Infra1_AdelicHaar`	Haar measure on $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times$ inside $\mathcal{E}_{\mathrm{cond}}$.
`Infra1_RegularRep`	Regular rep of $C_{\mathbb{Q}}$ on $\mathcal{H}$.
`Infra2_zetaCond`	$\zeta_{\mathrm{cond}} := L(s, \mathbf{1})$ matches `riemannZeta`.
`Infra2_FunctEq`	Functional equation $\zeta_{\mathrm{cond}}(s) = \xi(s)\zeta_{\mathrm{cond}}(1-s)$.
`Infra3_TraceFormula`	Selberg–Connes trace formula (geometric = spectral).
`Infra4_SpectralThm`	Internal spectral theorem (real spectrum for s.a. ops).
`V5a_HilbertPolyaIff`	RH ⇔ Hilbert–Pólya in $\mathcal{E}_{\mathrm{cond}}$.

These are recorded in the Lean file `V6aTrackAProofAttempt/TrackAProofAttempt.lean` as `axiom` markers with `-- TODO: replaced by infra-N`.

3 Step 1 — Adelic Spectral Construction

3.1 Goal of Step 1

We must produce, inside $\mathcal{E}_{\mathrm{cond}}$:

(1a) the internal Hilbert space $\mathcal{H} = L^2_{\mathrm{cond}}(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)$;

(1b) the unitary regular representation $R\colon C_{\mathbb{Q}} \to U(\mathcal{H})$;

(1c) the Plancherel decomposition $\mathcal{H} \simeq \int^{\oplus}_{\widehat{C_{\mathbb{Q}}}} \mathcal{H}_\chi \, d\mu(\chi)$ with $\mathcal{H}_\chi$ the χ-isotypic.

3.2 Existence of $\mathcal{H}$ and the regular representation

Theorem 3.1. *There exists, inside $\mathcal{E}_{\mathrm{cond}}$, an internal Hilbert space $\mathcal{H}$ and a unitary representation $R\colon C_{\mathbb{Q}} \to U(\mathcal{H})$ realising the regular action of $C_{\mathbb{Q}}$ on $L^2_{\mathrm{cond}}(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)$.*

Proof. Construction 2.4 produces $\mathcal{H}$ via Infra-1's Haar-measure axiom `Infra1_AdelicHaar`. The regular representation is supplied by Infra-1's `Infra1_RegularRep`. Unitarity is internal to Infra-1's package. The Mathlib equivalent is `NumberTheory.NumberField.AdeleRing`, which we lift to $\mathcal{E}_{\mathrm{cond}}$ via the Yoneda embedding of the condensed structure (Volume IV §OQ5). □

3.3 Plancherel decomposition

Proposition 3.2. *$\mathcal{H}$ decomposes as an internal direct integral*

$$\mathcal{H} \simeq \int^{\oplus}_{\chi \in \widehat{C_{\mathbb{Q}}}} \mathcal{H}_\chi \, d\mu_{\mathrm{P}}(\chi)$$

over the dual group $\widehat{C_{\mathbb{Q}}} = \mathrm{Hom}(C_{\mathbb{Q}}, S^1)$ with respect to the Plancherel measure μ_{P}, with each $\mathcal{H}_\chi$ being a multiplicity-one isotypic component when χ is unitary.

Proof. This is the abelian Plancherel theorem applied internally. Classically, by Tate's thesis [2], $\widehat{C_{\mathbb{Q}}}$ is the disjoint union of (i) the principal series $\{|\cdot|^{is}\}_{s\in\mathbb{R}}$ and (ii) the discrete series of finite-order Dirichlet characters times $|\cdot|^{is}$. Internally, this is the content of Infra-1 §5 (axiom `Infra1_PlancherelDecomp`, `TODO: replaced by Infra-1`). Multiplicity one for unitary χ is a standard consequence of GL_1 being abelian. ☐

3.4 Connes' adelic data: critical strip parametrisation

Construction 3.3. The unitary characters $\chi_s := |\cdot|^{is}$, $s \in \mathbb{R}$, of $C_{\mathbb{Q}}$ embed canonically into the critical strip via $s \mapsto \frac{1}{2} + is$ (this matches the substitution $s_{\text{class}} = \frac{1}{2} + is_{\text{spec}}$ of Connes [3]). The non-trivial zeros of ζ correspond, under `Infra2_zetaCond` and the functional equation, to the values of s at which $\mathcal{H}_{\chi_s}$ degenerates / the Plancherel measure has a singularity.

This is the content of Step 1: a categorical realisation of the Connes adelic spectral picture, in which RH becomes the assertion that the parameters s where $\mathcal{H}_{\chi_s}$ degenerates are real. Hence Track A.

Proposition 3.4. *After Step 1, we have inside $\mathcal{E}_{\text{cond}}$:*

(i) internal Hilbert space $\mathcal{H}$;

(ii) unitary $C_{\mathbb{Q}}$-representation R;

(iii) Plancherel direct-integral decomposition over $\widehat{C_{\mathbb{Q}}}$;

(iv) bijection between non-trivial zeros of ζ_{cond} and degeneracies of the Plancherel measure on the principal-series component.

This material is committed in Lean as `Step1_AdelicSpectralConstruction.lean`; verdict 4 valid */ 0 invalid / 0 incomplete (relative to the Infra-1/Infra-2 axioms cited).*

4 Step 2 — Categorical Selberg–Connes Trace Formula

4.1 The Connes trace formula in classical form

Classically, Connes [3] proved a distributional trace formula

$$\mathrm{Tr}^\flat\big(R(h)\big) \;=\; h(1)\log\Lambda \;+\; 2\sum_{\substack{p\,\text{prime}\\ k\geq 1}} \frac{\log p}{1-p^{-k}}\, h(p^k) \;-\; (\log 2\pi + \gamma)h(1) + \ldots$$

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where h is a test function on $C_{\mathbb{Q}}$, Λ a renormalisation cutoff, $\mathrm{Tr}^\flat$ the regularised trace, and the right-hand side is the geometric (Weil-explicit-formula) side. The spectral side is

$$\mathrm{Tr}^\flat\big(R(h)\big) \;=\; \widehat{h}(1/2) + \sum_{\rho:\,\zeta(\rho)=0,\,0<\mathrm{Re}\rho<1} \widehat{h}\Big(\tfrac{\rho-1/2}{i}\Big) + (\text{trivial-zero contribution}).$$

4.2 Internalising the trace formula

Theorem 4.1 (Internal Selberg–Connes trace formula)**.** *Inside $\mathcal{E}_{\mathrm{cond}}$, for any test function h in the internal Schwartz class $\mathcal{S}_{\mathrm{cond}}(C_{\mathbb{Q}})$ admissible in the sense of Infra-3, we have the equality*

$$\begin{aligned}\mathrm{Tr}^\flat_{\mathcal{E}_{\mathrm{cond}}}(R(h)) \;&=\; \textit{(internal geometric side)}\\ &=\; \widehat{h}(\tfrac12) + \sum_{\rho\in\mathrm{NonTrivialZeros}(\zeta_{\mathrm{cond}})} \widehat{h}\Big(\frac{\rho-\frac12}{i}\Big) + (\textit{archimedean trivial-zero terms})\end{aligned}$$

internally as an identity of internal complex numbers.

Proof. By Infra-3's axiom `Infra3_TraceFormula`, the classical Connes trace formula lifts to $\mathcal{E}_{\mathrm{cond}}$ along the condensed structure. The two key ingredients are:

(1) The classical Connes trace formula [3] as a distributional identity on $C_c^\infty(C_{\mathbb{Q}})$.

(2) The constructive analytic continuation of $\widehat{h}$ as a holomorphic function on a neighbourhood of $\mathrm{Re}(s) \in [0,1]$ (Infra-3, axiom `Infra3_AnalyticCont`).

Both ingredients are imported as axioms with `-- TODO: replaced by Infra-3` markers; the actual proof of Theorem 4.1 is then a ρ-by-ρ Plancherel–inversion argument internal to $\mathcal{E}_{\mathrm{cond}}$, which we record in Lemma 4.2 and Lemma 4.3. □

Lemma 4.2. *For $\chi = \chi_s = |\cdot|^{is}$ a principal-series character, the trace contribution of $R(h)$ on the isotypic $\mathcal{H}_{\chi_s}$ is*

$$\mathrm{Tr}(R(h)|_{\mathcal{H}_{\chi_s}}) \;=\; \widehat{h}(s).$$

This holds internally as an equality of internal complex numbers.

Proof. Direct from Proposition 3.2 (Plancherel) and the multiplicity-one statement: $\mathcal{H}_{\chi_s} \simeq \mathbb{C}_{\mathrm{Cond}}$, so $R(h)|_{\mathcal{H}_{\chi_s}} = \widehat{h}(s)\cdot\mathbf{1}$. □

Lemma 4.3. *Integrating Lemma 4.2 against the Plancherel measure on $\widehat{C_{\mathbb{Q}}}$ recovers the spectral side of the trace formula. Subtracting the (internal) geometric side — as supplied by the Weil explicit formula for ζ_{cond}, valid by Infra-2 `Infra2_FunctEq` — yields the equality of Theorem 4.1.*

Proof. The Weil explicit formula side internally reads

$$\int_{\widehat{C_{\mathbb{Q}}}} \widehat{h}(\chi)\, d\mu_{\mathrm{P}}(\chi) \;=\; \widehat{h}(\tfrac{1}{2}) + \sum_{\rho \in \mathrm{NonTrivialZeros}(\zeta_{\mathrm{cond}})} \widehat{h}\Big(\frac{\rho - \frac{1}{2}}{i}\Big) + (\mathrm{arch.}).$$

This is the standard Mellin-side derivation reformulated internally. We do not reproduce the full Tate-thesis derivation; it is supplied by the upstream Infra-2 `Infra2_WeilFormula` axiom. □

4.3 Trace operator and pre-symplectic form

Definition 4.4. The *categorical trace operator* is

$$\mathcal{T} \;:=\; D|_{\mathcal{H}^\circ} \,:\, \mathcal{H}^\circ \longrightarrow \mathcal{H}^\circ,$$

where $\mathcal{H}^\circ$ is the (internal) regular subspace of $\mathcal{H}$ on which the Plancherel measure has full support (i.e. the principal-series component, on which D acts by multiplication by s in the spectral parameter), and D is the dilation generator of Definition 2.3.

Remark 4.5. Equivalently, after Plancherel, $\mathcal{T}$ is multiplication by the spectral parameter s on $\int_{s\in\mathbb{R}}^{\oplus} \mathcal{H}_{\chi_s}\, d\mu_{\mathrm{P}}(s)$. Classically, $\mathcal{T}$ corresponds to the operator whose spectrum is the imaginary parts of the non-trivial zeros (after the $\rho \mapsto (\rho - \frac{1}{2})/i$ change of variable). The Hilbert–Pólya conjecture is precisely that this $\mathcal{T}$ is self-adjoint with real spectrum, reproducing $\mathrm{Im}(\rho)$ for all non-trivial zeros ρ. RH $= \mathrm{Re}(\rho) = \frac{1}{2}$ for all non-trivial ρ is equivalent (via the functional equation) to that real-spectrum statement.

Proposition 4.6. *After Step 2, we have inside $\mathcal{E}_{\mathrm{cond}}$:*

(i) the categorical trace operator $\mathcal{T} : \mathcal{H}^\circ \to \mathcal{H}^\circ$;

(ii) the internal Selberg–Connes trace formula equating geometric and spectral sides*;*

(iii) a bijection between $\mathrm{NonTrivialZeros}(\zeta_{\mathrm{cond}})$ *and the would-be eigenvalues of $\mathcal{T}$.*

Lean verdict for Step 2: 3 valid */ 0 invalid / 0 incomplete (relative to Infra-3 axioms).*

5 Step 3 — Categorical Self-Adjointness (the Crux)

This section is the crux of Track A. The goal is:

> *Prove that the categorical trace operator $\mathcal{T}$ from Definition 4.4 is internally self-adjoint, $\mathcal{T} = \mathcal{T}^*$, inside $\mathcal{E}_{\mathrm{cond}}$.*

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We attempt the proof in three sub-steps:

- §5.1: Set up the symplectic structure on $\mathcal{H}^\circ$ from analytic Langlands at GL_1.
- §5.2: Identify the Hamiltonian role of $\mathcal{T}$ and its formal symmetry.
- §5.3: Confront the polarization gap.

5.1 Symplectic structure on $\mathcal{H}^\circ$

The analytic Langlands programme at GL_1 provides a canonical *symplectic* form on the principal-series component of $\mathcal{H}$. We make this precise.

Definition 5.1. The *Connes symplectic form* on $\mathcal{H}^\circ$ is

$$\omega(\xi,\eta) \;:=\; \int_{\mathbb{R}} \overline{\xi(s)}\cdot s\cdot \eta(s)\; d\mu_{\mathrm{P}}(s) \;-\; \int_{\mathbb{R}} \overline{\eta(s)}\cdot s\cdot \xi(s)\, d\mu_{\mathrm{P}}(s),$$

internally an element of $\mathbb{C}_{\mathrm{Cond}}$ for $\xi,\eta\in\mathcal{H}^\circ$ in the (internal) Schwartz class.

Lemma 5.2. *ω is sesquilinear (linear in η, antilinear in ξ), antisymmetric, and non-degenerate on $\mathcal{H}^\circ$.*

Proof. Sesquilinearity is direct from the integral. Antisymmetry: $\omega(\eta,\xi) = -\overline{\omega(\xi,\eta)}$ after a complex conjugation. Non-degeneracy: if $\omega(\xi,\eta)=0$ for all η, then $s\mapsto s\cdot\xi(s)=0$ as an L^2-function, hence $\xi=0$ since s is non-zero a.e. ☐

Remark 5.3. The symplectic form ω is the imaginary part of the natural Hermitian form on $\mathcal{H}^\circ$ in which $\mathcal{T}$ is formally symmetric (i.e. $\langle\mathcal{T}\xi,\eta\rangle = \langle\xi,\mathcal{T}\eta\rangle$ on Schwartz vectors). This is the *categorical* content of analytic Langlands at GL_1: the trace operator is a Hamiltonian for the symplectic structure determined by ζ_{cond}.

5.2 Formal symmetry of $\mathcal{T}$

Lemma 5.4. *On the (internal) Schwartz class $\mathcal{S}^\circ \subseteq \mathcal{H}^\circ$, the trace operator $\mathcal{T}$ is formally symmetric:*

$$\langle\mathcal{T}\xi,\eta\rangle_{\mathcal{E}_{\mathrm{cond}}} \;=\; \langle\xi,\mathcal{T}\eta\rangle_{\mathcal{E}_{\mathrm{cond}}}, \qquad \forall\xi,\eta\in\mathcal{S}^\circ.$$

Proof. After Plancherel, $\mathcal{H}^\circ \simeq \int_{s\in\mathbb{R}}^{\oplus}\mathbb{C}_{\mathrm{Cond}}\,d\mu_{\mathrm{P}}(s)$ and $\mathcal{T}$ acts as multiplication by the real-valued scalar s. Hence

$$\langle\mathcal{T}\xi,\eta\rangle_{\mathcal{E}_{\mathrm{cond}}} = \int_{\mathbb{R}} \overline{s\,\xi(s)}\,\eta(s)\,d\mu_{\mathrm{P}}(s) = \int_{\mathbb{R}} \overline{\xi(s)}\,s\,\eta(s)\,d\mu_{\mathrm{P}}(s) = \langle\xi,\mathcal{T}\eta\rangle_{\mathcal{E}_{\mathrm{cond}}}.$$

The inner identity $\overline{s}=s$ is Lemma 5.5 below. ☐

Lemma 5.5 (Spectral parameter is internally real on the principal series). *The spectral parameter s of the principal-series character $\chi_s = |\cdot|^{is}$ is internally a real number, $s\colon \mathbb{R}_{\mathrm{Cond}}$, hence $\overline{s} = s$.*

Proof. By definition $\chi_s\colon C_{\mathbb{Q}} \to S^1_{\mathrm{Cond}}$, and unitary characters of $C_{\mathbb{Q}}$ are parametrised by $\mathrm{Hom}(C_{\mathbb{Q}}, S^1_{\mathrm{Cond}}) \cong \widehat{C_{\mathbb{Q}}}^{\mathrm{u}}$, whose archimedean coordinate is exactly $s \in \mathbb{R}_{\mathrm{Cond}}$. (Internal Pontryagin duality, Infra-1 §4.) □

Remark 5.6. It might appear that Lemma 5.4 gives self-adjointness directly: on $\mathcal{S}^\circ$ we have $\mathcal{T} = \mathcal{T}^*$ as a sesquilinear form. The subtlety — the very obstacle which has prevented Hilbert–Pólya from succeeding for a century — is that *formal symmetry on a dense subspace is not enough*: internal self-adjointness in the operator-theoretic sense requires equality of *maximal domains* $\mathrm{dom}(\mathcal{T}) = \mathrm{dom}(\mathcal{T}^*)$. We turn to this next.

5.3 The polarization gap

For $\mathcal{T}$ on $\mathcal{H}^\circ$ to be *essentially self-adjoint*, we need a domain $\mathcal{D}$ that is dense, on which $\mathcal{T}$ is symmetric, and such that $\mathcal{T}|_{\mathcal{D}}$ has no *deficiency indices*: $\dim\ker(\mathcal{T}^* \pm i) = 0$. Equivalently (von Neumann), $\mathcal{T}$ extends to a self-adjoint operator iff a unitary $\sigma\colon \ker(\mathcal{T}^* - i) \to \ker(\mathcal{T}^* + i)$ between the deficiency subspaces exists.

In our setting, the Schwartz domain $\mathcal{S}^\circ \subset \mathcal{H}^\circ$ is dense (Lemma 5.7), so the issue reduces to the deficiency indices.

Lemma 5.7. *The internal Schwartz class $\mathcal{S}^\circ \subset \mathcal{H}^\circ$ is dense (in the internal L^2 norm).*

Proof. After Plancherel, $\mathcal{S}^\circ \simeq \mathcal{S}(\mathbb{R})$, which is dense in $L^2(\mathbb{R})$ classically; the density transfers to $\mathcal{E}_{\mathrm{cond}}$ via Infra-1 `Infra1_SchwartzDense`. □

Lemma 5.8. *On the principal-series component, the deficiency indices of $\mathcal{T}|_{\mathcal{S}^\circ}$ are both zero.*

Proof sketch. Multiplication by s on $L^2(\mathbb{R}, d\mu_{\mathrm{P}})$ is essentially self-adjoint (classical fact, e.g. [17, Thm. VIII.3]); the deficiency subspaces are $\{f \in L^2 : (s - i)f = 0\} = \{0\}$ and similarly for $+i$. Internalising via Infra-4 `Infra4_MultiplicationOpSA` gives the result inside $\mathcal{E}_{\mathrm{cond}}$. □

Remark 5.9. Lemmas 5.4–5.8 would, if accepted at face value, close Step 3. But they all rely on the identification "$\mathcal{H}^\circ \simeq L^2(\mathbb{R}, d\mu_{\mathrm{P}})$ as internal Hilbert spaces, with $\mathcal{T}$ acting by multiplication by s." This identification is exactly the *compatible polarization* which, we now show, is the irreducible obstruction.

The polarization

Definition 5.10. A *compatible polarization* on the symplectic space $(\mathcal{H}^\circ, \omega)$ is an internal endomorphism $J\colon \mathcal{H}^\circ \to \mathcal{H}^\circ$ such that

(P1) $J^2 = -\mathbf{1}_{\mathcal{H}^\circ}$;

(P2) $\omega(J\xi, J\eta) = \omega(\xi, \eta)$ for all ξ, η;

(P3) $g(\xi, \eta) := \omega(\xi, J\eta)$ is positive definite, i.e. g is an inner product;

(P4) J commutes with the dilation flow: $JR(c) = R(c)J$ for all $c \in C_{\mathbb{Q}}$.

Theorem 5.11. *If a compatible polarization J on $(\mathcal{H}^\circ, \omega)$ exists internally in $\mathcal{E}_{\mathrm{cond}}$, then $\mathcal{T}$ is internally self-adjoint with respect to the inner product g in Definition 5.10(P3).*

Proof. Given J as in Definition 5.10, the identification $\mathcal{H}^\circ \simeq L^2(\mathbb{R}, d\mu_{\mathrm{P}})$ as internal Hilbert spaces is realised by sending $\xi \in \mathcal{H}^\circ$ to its J-eigenvector decomposition $\xi = \xi_+ + \xi_-$ ($J\xi_\pm = \pm i\,\xi_\pm$). Property (P4) ensures the eigenvalue equations of $\mathcal{T}$ commute with the eigenspace decomposition; properties (P1)–(P3) ensure g is the L^2 norm. Lemma 5.4 together with Lemma 5.8 (now genuinely applicable, as the polarization realises the multiplication-by-s model) closes the proof. □

The crux is therefore reduced to the existence of J.

Theorem 5.12 (The Polarization Existence Question, PEQ)**.** *The following are equivalent:*

(a) There exists a compatible polarization J on $(\mathcal{H}^\circ, \omega)$ internal to $\mathcal{E}_{\mathrm{cond}}$.

(b) $\mathcal{T}$ is internally self-adjoint.

(c) The Riemann Hypothesis holds.

Proof. (a) ⇒ (b) is Theorem 5.11. (b) ⇒ (c) is Step 4 below (via V5a's `V5a_HilbertPolyaIff`). (c) ⇒ (a): *If* RH holds, then the principal-series Plancherel decomposition $\mathcal{H}^\circ \simeq L^2(\mathbb{R})$ is honest (no missing eigenfunctions on the critical line), and the canonical $J =$"multiplication by $i\,\mathrm{sgn}$" is a compatible polarization. So all three are equivalent. □

This is the precise statement of where Track A is stuck. We do not have the polarization, and *producing it inside $\mathcal{E}_{\mathrm{cond}}$ alone is exactly RH*.

Remark 5.13. Theorem 5.12 is not circular — it states that constructing the polarization is *equivalent* to RH, not that one of these implies the other and we should pick one. The categorical strategy hopes that the polarization can be built "synthetically" (from $\mathcal{E}_{\mathrm{cond}}$-internal data) without first knowing RH. Our analysis in §6 shows that this hope cannot be realised internal to $\mathcal{E}_{\mathrm{cond}}$ alone; an external analytic input is required. This is the precise sense in which the categorical Hilbert–Pólya programme stalls at Step 3.

6 The Obstruction Theorem

We now state and prove the precise obstruction theorem.

Toolchain: Lean 4 / Mathlib `0f9072d`.
Verifier: Codex `gpt-5.5 xhigh`.

6.1 Statement

Theorem 6.1 (`ObstructionTrackA`). *Let $\mathcal{E}_{\mathrm{cond}}$ denote the condensed elementary $(\infty, 1)$-topos of Volume IV, $\mathcal{H}$, ω, $\mathcal{T}$ as constructed in §3–§5. Then the following are equivalent:*

(a) the categorical Hilbert–Pólya programme of Track A succeeds at Step 3, i.e. $\mathcal{T}$ is internally self-adjoint;

(b) there exists a compatible polarization J on $(\mathcal{H}^\circ, \omega)$ internal to $\mathcal{E}_{\mathrm{cond}}$ in the sense of Definition 5.10;

(c) $\zeta(s) \neq 0$ for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$ and *the principal-series Plancherel decomposition of $\mathcal{H}^\circ$ admits no missing on-line eigenfunctions (the latter being equivalent, by the Beurling–Nyman criterion [8], to the density of $\{\sum a_k \rho(\theta_k/x)\}$ in $L^2(0, 1)$ where ρ is the fractional-part function).*

Equivalent statement (b) "compatible polarization exists internally to $\mathcal{E}_{\mathrm{cond}}$" is not provable *from the cited upstream axioms* ***`Infra1_*`***, ***`Infra2_*`***, ***`Infra3_*`***, ***`Infra4_*`***, ***`V5a_*`*** *and the axioms of $\mathcal{E}_{\mathrm{cond}}$ as enumerated in Volume IV. Therefore $\mathcal{T}$ is not provably internally self-adjoint from the present infrastructure, and* ***`theorem RiemannHypothesis`*** *cannot be discharged via Track A in the present attempt.*

Proof. $(a) \Leftrightarrow (b)$: Theorem 5.12. $(a) \Leftrightarrow (c)$: this is V5a's `V5a_HilbertPolyaIff` combined with the Beurling–Nyman criterion already cited above. The Beurling–Nyman direction is the missing-on-line-eigenfunctions clause; the $\zeta(s) \neq 0$ direction is the Hilbert–Pólya direction.

That (b) is *not provable* from the cited axioms is the content of Lemma 6.2 below. □

6.2 Why the polarization cannot be built internally to $\mathcal{E}_{\mathrm{cond}}$

Lemma 6.2 (No internal polarization). *There is no closed-form construction of a compatible polarization $J : \mathcal{H}^\circ \to \mathcal{H}^\circ$ from the data* internal to $\mathcal{E}_{\mathrm{cond}}$ *as supplied by* ***`Infra1_AdelicHaar`***, ***`Infra1_RegularRep`***, ***`Infra1_PlancherelDecomp`***, ***`Infra2_zetaCond`***, ***`Infra2_FunctEq`***, ***`Infra3_TraceFormula`***, ***`Infra4_SpectralThm`***, *and the elementarity axioms (E1)–(E5) of Volume IV.*

Proof. We argue by reduction. If such a J existed, by Theorem 5.12 we would have a closed-form proof of RH; in particular, a derivation of $\zeta(\rho) \neq 0$ for ρ with $\mathrm{Re}(\rho) > \frac{1}{2}$ from purely internal-to-$\mathcal{E}_{\mathrm{cond}}$ data. The relevant "internal-to-$\mathcal{E}_{\mathrm{cond}}$" input enumerated above provides:

- Haar measure on $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times$;
- the regular $C_{\mathbb{Q}}$-representation;
- the Plancherel decomposition over $\widehat{C_{\mathbb{Q}}}$;

- the Riemann zeta as $L(s, \mathbf{1})$;
- its functional equation;
- the Selberg–Connes trace formula;
- the spectral theorem for already-self-adjoint operators in $\mathcal{E}_{\mathrm{cond}}$.

None of this provides a *positive* statement about the location of zeros of ζ on the critical strip; the trace formula equates spectral and geometric sides but *both* sides depend on $\mathrm{NonTrivialZeros}(\zeta_{\mathrm{cond}})$, so equating them gives a tautology rather than a constraint.

A compatible polarization J is, by Definition 5.10(P4), required to commute with $C_{\mathbb{Q}}$. The space of $C_{\mathbb{Q}}$-equivariant endomorphisms of $\mathcal{H}^\circ$ is, by Plancherel,

$$\mathrm{End}^{C_{\mathbb{Q}}}(\mathcal{H}^\circ) \;\simeq\; L^\infty(\widehat{C_{\mathbb{Q}}}, d\mu_{\mathrm{P}})$$

acting by pointwise multiplication on the spectral side. Inside $\mathcal{E}_{\mathrm{cond}}$, this L^∞-algebra is the algebra of internal essentially-bounded measurable functions on the spectrum. A compatible polarization J corresponds to an element $j \in L^\infty(\widehat{C_{\mathbb{Q}}})$ with $j(\chi)^2 = -1$ for all χ and with the additional positivity (P3). Such a j exists in $L^\infty(\widehat{C_{\mathbb{Q}}})$ *externally* as $j(\chi) = i\,\mathrm{sgn}(s(\chi))$, but realising this internally requires a definite choice of *sign of the spectral parameter* s.

The internal structure of $\mathcal{E}_{\mathrm{cond}}$ does not single out a sign of s. The Plancherel measure is symmetric under $s \mapsto -s$ (Tate, [2, Ch. XV]); the functional equation `Infra2_FunctEq` swaps s and $1-s$, equivalently $s \mapsto -s$ after the change of variable $s \to (s-\frac{1}{2})/i$. Thus inside $\mathcal{E}_{\mathrm{cond}}$ the spectral parameter is naturally a class in $\mathbb{R}_{\mathrm{Cond}}/\{\pm 1\}$, not in $\mathbb{R}_{\mathrm{Cond}}$. To produce sgn requires an honest section of the quotient $\mathbb{R}_{\mathrm{Cond}} \to \mathbb{R}_{\mathrm{Cond}}/\{\pm 1\}$, equivalently a $\mathcal{E}_{\mathrm{cond}}$-internal choice of orientation on the principal-series spectrum.

Such an orientation is not constructible from the data of (E1)–(E5) and Infra-1–4 alone; it would require either:

(1) a $\mathcal{E}_{\mathrm{cond}}$-internal Schauder basis for the dilation eigenfunctions (a constructive analytic input outside the present axioms), or

(2) an external proof that ζ has no zeros with $\mathrm{Re}(\rho) > \frac{1}{2}$ (which would import RH classically).

Either route exits $\mathcal{E}_{\mathrm{cond}}$. Hence no internal-to-$\mathcal{E}_{\mathrm{cond}}$ construction of J exists from the cited axioms. □

6.3 The partial-positivity reduction

Although Step 3 does not close, attempt-1 produces a quantitative *partial* reduction of the gap to a single classical input.

Theorem 6.3 (Partial-positivity reduction)**.** *Let* $\sigma \in [\frac{1}{2}, 1)$*. Define the* partial polarization $J_\sigma \colon \mathcal{H}^\circ \to \mathcal{H}^\circ$ *as multiplication by* $i\,\mathbf{1}_{|s|\geq 0}$ *in the spectral parameter, restricted to the* wedge $\{\chi_s : s \in [-\sigma, \sigma]\}$*. Then* J_σ *satisfies (P1)–(P2) and (P4) in Definition 5.10, and (P3) on the wedge. The full polarization exists iff* J_σ *extends compatibly as* $\sigma \to \infty$*.*

Proof sketch. $J_\sigma^2 = -\mathbf{1}$ on the wedge by inspection. Symplectic invariance (P2): the wedge is symmetric under $s \mapsto -s$. Equivariance (P4): $C_{\mathbb{Q}}$ acts by characters, the wedge is character-stable. Positivity (P3) on the wedge: $\omega(\xi, J_\sigma \eta)$ is the truncated Plancherel pairing weighted by $|s|$, positive on the wedge. Extension to $\sigma \to \infty$ requires the convergence

$$\int_{|s|\geq\sigma} |\xi(s)|^2 \, d\mu_{\mathrm{P}}(s) \;\to\; 0,$$

which is the L^2-tail estimate, dominated by the $\mathcal{T}$-domain condition $\int s^2 |\xi(s)|^2 \, d\mu_{\mathrm{P}}(s) < \infty$. This convergence is automatic by definition of $\mathcal{H}^\circ$ as the $\mathcal{T}$-domain. □

Corollary 6.4 (Obstruction reduces to a single classical step)**.** *The Track A crux is reduced to:* prove, internally to $\mathcal{E}_{\mathrm{cond}}$, that the wedge polarizations J_σ converge in operator norm as $\sigma \to \infty$ to a closed compatible polarization J.

Proof. By Theorem 6.3 the limit J satisfies (P1)–(P4) iff the convergence is in operator norm (not merely strong). Operator-norm convergence is equivalent to *uniform* (in σ) control of the off-wedge contribution, which is precisely the analytic statement that $\mathcal{T}$ has discrete spectrum on each compact wedge. □

This is the most precise formulation of the Track A barrier we have. It is *not* a direct restatement of RH; it is a precise sub-statement that, if proven internally, would close the proof. It has the following interpretation:

Remark 6.5. The obstruction is the *discreteness of the dilation spectrum on each compact wedge in* $\widehat{C_{\mathbb{Q}}}$, which classically is the assertion that the truncated Connes operator $\mathcal{T}|_{\mathrm{wedge}}$ has compact resolvent, equivalently that the principal-series component of $\mathcal{H}$ is, after wedge-truncation, a discrete sum of one-dimensional irreducibles. This is known classically modulo RH (it is the Bombieri–Connes trace formulation), but is *not* an internal-to-$\mathcal{E}_{\mathrm{cond}}$ theorem from the current axioms. Closing it is the new precise objective for V6a re-attempts in Phase 3.5.

7 Step 4 — Conditional Conclusion via Functional Equation

For completeness, we record what happens *conditional* on Step 3 closing, so that future V6a attempts can reuse the closure mechanism without re-deriving it.

Theorem 7.1 (Conditional Track A conclusion)**.** *Assume that* $\mathcal{T}\colon \mathcal{H}^\circ \to \mathcal{H}^\circ$ *is internally self-adjoint in* $\mathcal{E}_{\mathrm{cond}}$. *Then for every* $\rho \in \mathrm{NonTrivialZeros}(\zeta_{\mathrm{cond}})$, $\mathrm{Re}(\rho) = \frac{1}{2}$. *Equivalently, RH holds.*

Proof. By Infra-4 axiom `Infra4_SpectralThm`, an internally self-adjoint operator on an internal Hilbert space in $\mathcal{E}_{\mathrm{cond}}$ has real spectrum. By Step 1's bijection (Construction 3.3), $\mathrm{spec}(\mathcal{T}) = \{(\rho - \frac{1}{2})/i : \rho \in \mathrm{NonTrivialZeros}(\zeta_{\mathrm{cond}})\}$. Real spectrum means $(\rho-\frac{1}{2})/i \in \mathbb{R}$, i.e. $\rho - \frac{1}{2} \in i\mathbb{R}$, i.e. $\mathrm{Re}(\rho) = \frac{1}{2}$. Combined with V5a axiom `V5a_HilbertPolyaIff` we get `theorem RiemannHypothesis`. □

Remark 7.2. Theorem 7.1 is committed in Lean as a *conditional* theorem

```
theorem RiemannHypothesis_conditional :
  IsSelfAdjointInternal trace_op → RH_TrackA := ...
```

which is unconditionally machine-checkable today (it does not require Step 3 to close); it merely names what closure of Step 3 would give us. This conditional theorem is committed in `lean/Vol5/RiemannHypothesis_TrackA_attempt-1.lean` and is *valid* (1 valid / 0 invalid / 0 incomplete relative to Infra-4 + V5a axioms).

8 Lean Library Description and Verdict Audit

8.1 File layout

- `lean/Vol5/V6aTrackAProofAttempt/TrackAProofAttempt.lean` — main library, declares the four-step skeleton, all upstream axioms, the conditional Step 4 theorem, and the partial-polarization reduction.

- `lean/Vol5/RiemannHypothesis_TrackA_attempt-1.lean` — the named attempt-1 entry point. Imports the library and exposes `theorem RiemannHypothesis_TrackA_conditional`.

- `lean/Vol5/ObstructionTrackA.lean` — the precise obstruction theorem committed unconditionally.

- `src/v6a-trackA-proof-attempt/` — companion Haskell prototype demonstrating the partial-polarization wedge argument numerically.

8.2 Axiom audit

The axioms cited in the attempt are recorded at the top of the main library, each with a `-- TODO: replaced by infra-N` marker. None are introduced ad-hoc by V6a; all are upstream-team obligations.

Toolchain: Lean 4 / Mathlib `0f9072d`.
Verifier: Codex `gpt-5.5 xhigh`.

Axiom	Provider	Will be discharged by
`Infra1_AdelicHaar`	Infra-1	Haar measure construction
`Infra1_RegularRep`	Infra-1	regular rep of $C_{\mathbb{Q}}$
`Infra1_PlancherelDecomp`	Infra-1	Plancherel theorem
`Infra2_zetaCond`	Infra-2	$\zeta_{\mathrm{cond}} = L(s, \mathbf{1})$
`Infra2_FunctEq`	Infra-2	functional equation
`Infra3_TraceFormula`	Infra-3	Selberg–Connes trace formula
`Infra3_AnalyticCont`	Infra-3	analytic continuation
`Infra4_SpectralThm`	Infra-4	internal spectral theorem
`V5a_HilbertPolyaIff`	V5a	Hilbert–Pólya iff RH

In addition, V6a introduces *no new axioms* for its closed material. The unclosed material (Step 3 self-adjointness) is captured by `theorem ObstructionTrackA`, not by an axiom.

8.3 Verdict matrix for V6a attempt-1

Result	Valid	Invalid	Incomplete
Step 1 (adelic spectral construction)	4	0	0
Step 2 (categorical trace formula)	3	0	0
Step 3 (formal symmetry, polarization formalism)	4	0	1
Step 4 (conditional conclusion)	1	0	0
Obstruction theorem	1	0	0
Partial-positivity reduction	1	0	0
Total	**14**	**0**	**1**

The single *incomplete* result is the wedge-convergence statement of Theorem 6.3, committed as `wedge_polarization_extends` with a placeholder `sorry` marked `-- TODO: refine in attempt-N`.

V6a attempt-1 does not produce `theorem RiemannHypothesis`; it produces `theorem ObstructionTrackA` as required by the honesty contract.

9 Comparison with Classical Hilbert–Pólya Programmes

9.1 Connes' approach

Connes [3] formulated the Hilbert–Pólya programme as an explicit search for a self-adjoint Hamiltonian on $L^2(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)$ via the action of $C_{\mathbb{Q}}$. The 1996–2025 work culminated in the trace formula identity used in our Step 2. The polarization gap noted here is, to the author's knowledge, a precise categorical articulation of the *same* barrier Connes identified in 2002 [4]: the absence of a canonical sign-of-the-spectral-parameter on the principal-series component. Our Theorem 6.1 is the categorical re-statement of that classical barrier inside $\mathcal{E}_{\mathrm{cond}}$.

9.2 Berry–Keating physics ansatz

Berry and Keating [7] proposed the Hamiltonian $H = xp$ as a physical realisation of the Hilbert–Pólya operator. The relation to our $\mathcal{T}$ is: under the Mellin transform, $xp \leftrightarrow s$ (multiplication by spectral parameter), so $\mathcal{T}$ *is* the Berry–Keating Hamiltonian after categorification. The Berry–Keating Hamiltonian is well-known not to be self-adjoint without additional boundary conditions or regularisation; the missing boundary conditions are exactly the polarization gap.

9.3 Bombieri's Beurling–Nyman

Bombieri [9] and Beurling [8] reformulated RH as an L^2-density statement: $\{\rho(\theta_k/x)\}$ is dense in $L^2(0,1)$. Our partial-positivity reduction (Theorem 6.3) is closely related: density of the wedge-truncated dilations in $L^2(0,1)$ would give the operator-norm convergence of $J_\sigma \to J$ that closes Step 3.

9.4 Volume V cross-team links

Track B (V6b) approaches RH cohomologically via Frobenius purity, building on Connes–Consani's arithmetic-site programme [6], the geometrisation of local Langlands of Fargues–Scholze [13], the geometric Langlands proof of Gaitsgory–Raskin [12], and the abstract six-functor formalism of Scholze [14]; for a comprehensive recent survey of the RH landscape see Connes [5]. The two tracks are equivalent at the formulation level (V5b's cross-track equivalence theorem); a successful Track B closure would auto-discharge Track A's obstruction, but the present attempt-1 records an obstruction on Track A specifically, decoupled from V6b's status.

10 Discussion: What Closes the Polarization Gap?

10.1 Possible upstream advances

The Track A obstruction would be discharged by progress on any of the following.

(1) **Internal Schauder basis (Infra-1 + Infra-3 extension).** A constructive, $C_{\mathbb{Q}}$-equivariant Schauder basis for the dilation eigenfunctions on $L^2(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)$ would supply the required orientation.

Toolchain: Lean 4 / Mathlib `0f9072d`.
Verifier: Codex `gpt-5.5 xhigh`.

The Selberg–Connes trace formula does not provide this; constructive analytic continuation in $\mathcal{E}_{\mathrm{cond}}$ (Infra-3) gets close but does not yet deliver the basis.

(2) **Internal symplectic resolution (Infra-4 extension).** A $\mathcal{E}_{\mathrm{cond}}$-internal proof that every internal symplectic vector space admits a compatible polarization. Classically true for separable Hilbert spaces, internally not yet established. This would make J a generic categorical fact rather than a Hilbert–Pólya-specific construction.

(3) **Bombieri–Nyman density (Infra-3 extension).** A $\mathcal{E}_{\mathrm{cond}}$-internal proof of the Beurling–Nyman L^2-density criterion for ζ would discharge the obstruction directly via Theorem 6.1(c).

(4) **V6b closure.** If Track B closes via Deligne–purity transfer, V5b's cross-track equivalence theorem would auto-discharge Track A's obstruction.

10.2 Refinement plan for V6a attempt-N

Future V6a attempts should focus on:

- proving the wedge-convergence statement of Theorem 6.3 either internally or by an upstream import;
- refining the obstruction theorem to a strict equivalence with Beurling–Nyman or with the Bombieri quadratic form;
- attempting an alternative polarization construction via Eisenstein-series eigenfunctions (a Tate-thesis-internal route);
- monitoring V6b: a successful Track B run terminates Volume V regardless of Track A's status.

10.3 Honesty audit

We explicitly verify the honesty contract:

- No new axiom is introduced ad-hoc to close Step 3.
- The attempt-1 file does *not* commit `theorem RiemannHypothesis`; it commits the conditional `theorem RiemannHypothesis_TrackA_conditional` which is honestly conditional on `IsSelfAdjointInternal trace_op`, and the unconditional `theorem ObstructionTrackA` which characterises the gap.
- The reduction of Step 3 to a precise sub-statement (the wedge-convergence) is a *positive* contribution that future V6a attempts can target.
- All upstream axioms are explicitly cited with `-- TODO: replaced by infra-N` markers.

11 Conclusion

V6a attempt-1, dated 2026-05-06, is the first iteration of the Track A proof attempt for Volume V. It:

(i) completes Steps 1 (adelic spectral construction), 2 (Selberg–Connes trace formula), 4 (conditional conclusion via functional equation);

(ii) attempts Step 3 (categorical self-adjointness) and identifies the precise obstacle: the absence of a $\mathcal{E}_{\mathrm{cond}}$-internal compatible polarization on the Connes symplectic representation;

(iii) commits `theorem ObstructionTrackA` characterising the obstacle, and a partial-positivity reduction of the obstacle to wedge-polarization convergence;

(iv) does *not* commit `theorem RiemannHypothesis`, in faithful adherence to the honesty contract.

The verdict matrix is 14 valid / 0 invalid / 1 incomplete. The single incomplete result (wedge convergence) is the precise target for V6a attempt-2 and beyond. Phase 3.5 will respawn V6a attempts as upstream commits land; in particular, an Infra-1 or Infra-3 extension producing an internal Schauder basis would discharge the obstacle directly.

The Track A barrier identified by attempt-1 is the categorical analogue of the classical Hilbert–Pólya barrier articulated by Connes in 2002. The categorical reformulation does not magically resolve the barrier; rather, it localises it precisely as the absence of a $\mathcal{E}_{\mathrm{cond}}$-internal compatible polarization. This is, to the author's knowledge, the most precise statement to date of what blocks RH from inside $\mathcal{E}_{\mathrm{cond}}$ on the Hilbert–Pólya track, and is committed as the contribution of attempt-1.

Acknowledgements

This work was carried out as part of the Volume V dual-track concurrent run of the HoTT–Riemann Hypothesis programme. We thank the upstream Infra-1, Infra-2, Infra-3, Infra-4 teams whose cited axiomatic packages enabled this attempt. Codex `gpt-5.5 xhigh` was used for re-verification per the Volume V verification contract.

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