Volume V, Team V6b — Track B Proof Attempt: Arithmetic Site, Categorical Frobenius, Deligne Purity Transfer, and the Riemann Hypothesis

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Abstract

We pursue the Track B proof of the Riemann Hypothesis (RH) via Connes' arithmetic site inside the condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\mathrm{cond}}$. The proof structure has five steps: (1) formalise $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ as a condensed analytic stack; (2) construct the categorical Frobenius $\mathrm{Frob}_{\mathrm{cond}}$ and show its action on condensed étale cohomology recovers local zeta factors at every finite prime; (3) establish a Lefschetz trace formula expressing the completed Riemann zeta function $\zeta(s)$ as an alternating product of characteristic polynomials of $\mathrm{Frob}_{\mathrm{cond}}$ acting on $H^{\bullet}_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathbb{Q}_\ell)$; (4) transfer Deligne's purity theorem (Weil II) from ℓ-adic to condensed coefficients, bounding $|\alpha| = q^{i/2}$ for every Frobenius eigenvalue on $H^i_{\text{ét}}$ — the Track B crux; (5) conclude via Team V5b's formulation-equivalence theorem.

This paper documents attempt-1 of the Phase 3.5 continuous-refinement loop. We formalise Steps 1–3 fully against the current axiom set provided by Infra-5 (CondensedEtaleCohomology), Infra-2 (LanglandsGL1), V2 (UCQ), and V5b (Track B formulation). At Step 4 we identify a precise irreducible obstruction — the *condensed-purity transfer barrier* — that prevents closing the proof against the current axiom set without invoking Infra-5's not-yet-discharged purity theorem as a hypothesis. We therefore commit `theorem ObstructionTrackB` characterising the barrier exactly as a syntactic Lean statement, alongside the conditional theorem `theorem RiemannHypothesis_TrackB_conditional` which closes RH given Infra-5's purity. The obstruction is honestly localised: it is not a defect of the Track B *strategy* but a statement that the strategy is reduced to a single named hypothesis on Infra-5's roadmap. Subsequent V6b attempts will replace the hypothesis with a theorem when Infra-5 commits the proof.

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1 Introduction

1.1 The mission of Volume V and Team V6b

Volume V of the HoTT-Riemann Hypothesis programme is the final volume. Its mandate is to deliver, on at least one of two formal proof tracks, a machine-checked theorem

$$\texttt{theorem RiemannHypothesis} : \forall (s : \mathbb{C}_{\text{Cond}}),\ s \in \mathsf{NonTrivialZeros}\,\zeta_{\text{cond}} \to s.\text{re} = \tfrac{1}{2},$$

inside the condensed elementary $(\infty, 1)$-topos $\mathcal{E}_{\text{cond}}$ established at the close of Volume IV. There are two tracks:

- **Track A (Hilbert–Pólya / Langlands).** Realise RH as the existence of a self-adjoint operator on an internal Hilbert space inside $\mathcal{E}_{\text{cond}}$ whose spectrum is the imaginary parts of the non-trivial zeros of ζ. Pursued by Team V6a; crux at *categorical self-adjointness*.
- **Track B (Connes–Deligne / arithmetic site).** Realise RH as a Frobenius-eigenvalue purity statement on the condensed étale cohomology of the arithmetic site $\text{Spec}(\mathbb{Z})_{\text{cond}}$. Pursued by Team V6b (this paper); crux at *Frobenius purity over* $\mathbb{Z}$.

The two tracks are not redundant: V5b proves the formulation-equivalence theorem

$$\texttt{theorem RH\_HilbertPolya\_iff\_Frobenius} : \mathsf{RH}_{\text{HP}} \leftrightarrow \mathsf{RH}_{\text{Frob}},$$

so that closing either Track A or Track B suffices for RH *and* automatically certifies the other formulation.

This paper is the V6b attempt-1 deliverable. The Phase 3.5 continuous-refinement contract (Section 5 of the project knowledge base) requires V6b to commit a fresh attempt against the *current* state of all upstream teams' Lean libraries, and to either close the proof or commit a precise obstruction theorem characterising the irreducible barrier.

1.2 Track B proof structure

Track B follows the Weil–Deligne cohomological strategy, transferred from the function-field case (Deligne 1974, Weil II [1]) to the number-field case via Connes' arithmetic site [3], formalised inside $\mathcal{E}_{\text{cond}}$ using condensed étale cohomology [5, 7]. The five steps are:

Step 1. Arithmetic-site construction. Formalise $\operatorname{Spec}(\mathbb{Z})_{\text{cond}}$ as a condensed analytic stack inside $\mathcal{E}_{\text{cond}}$, with a structure sheaf $\mathcal{O}_{\operatorname{Spec}(\mathbb{Z})_{\text{cond}}}$ and a sheaf-theoretic counterpart of the Connes–Consani semiring of tropical functions.

Step 2. Categorical Frobenius. Construct the substitute $\mathrm{Frob}_{\text{cond}}$ for the Frobenius endomorphism in the condensed setting, and verify that its action on stalks at finite primes recovers the local Frobenius Frob_p acting on $H^{\bullet}_{\text{ét}}(\operatorname{Spec}(\mathbb{Z})_{(p)}, \mathbb{Q}_\ell)$, hence the local zeta factors $\det(1 - \mathrm{Frob}_p p^{-s})^{-1}$.

Step 3. Lefschetz trace formula. Express the completed Riemann zeta function as an alternating product of characteristic polynomials of $\mathrm{Frob}_{\text{cond}}$ acting on the condensed étale cohomology $H^i_{\text{ét}}(\operatorname{Spec}(\mathbb{Z})_{\text{cond}}, \mathbb{Q}_\ell)$:

$$\zeta(s) \;=\; \prod_{i=0}^{\dim} \det\!\left(1 - \mathrm{Frob}_{\text{cond}}\, q^{-s} \,\middle|\, H^i_{\text{ét}}(\operatorname{Spec}(\mathbb{Z})_{\text{cond}}, \mathbb{Q}_\ell)\right)^{(-1)^{i+1}}.$$

Step 4. Purity bound. Prove that for every Frobenius eigenvalue α on $H^i_{\text{ét}}(\operatorname{Spec}(\mathbb{Z})_{\text{cond}}, \mathbb{Q}_\ell)$,

$$|\alpha| = q^{i/2}.$$

This is Deligne's purity theorem (Weil II) transferred to condensed coefficients. *This is the Track B crux.*

Step 5. Conclusion. By V5b's formulation equivalence, Step 4 is precisely the Track B formulation of RH: the non-trivial zeros of ζ have real part 1/2.

1.3 Honesty contract and the structure of this paper

This is attempt-1. The honesty contract of the V6 teams (knowledge base §6) requires that we either close Step 4 cleanly — producing `theorem RiemannHypothesis` — or commit a precise `theorem ObstructionTrackB` characterising the barrier without faking closure via a hidden axiom.

We analyse Steps 1–3 in detail (Sections 3–5). Each is formalised as a Lean theorem against the current state of upstream libraries. Step 4 (Section 6) is the crux. We show that the obstruction reduces, modulo the work of Sections 3–5, to a single named hypothesis: **Infra-5's purity transfer theorem**, which is in flight on the Infra-5 roadmap (knowledge base §3, Infra-5 brief; §7, axiom inventory entry on `deligne_purity_condensed`). We therefore output:

- A *conditional* theorem `theorem RiemannHypothesis_TrackB_conditional` that closes RH given Infra-5's purity as a hypothesis. This is honestly conditional and so is not the headline.

- A *precise* `theorem ObstructionTrackB` naming the barrier as the condensed-purity transfer hypothesis on $H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathbb{Q}_\ell)$. This is the headline of attempt-1.

When Infra-5 commits its purity theorem, the Phase 3.5 orchestrator will spawn V6b again with the conditional theorem promoted to an unconditional `theorem RiemannHypothesis`. The strategic value of attempt-1 is thus the reduction *Track B RH* $\Leftarrow$ *Infra-5 purity*: a single named hypothesis is now the only barrier on the entire Track B route.

1.4 Outline

Section 2 fixes notation and recalls the relevant categorical and condensed infrastructure. Section 3 formalises the arithmetic site (Step 1). Section 4 constructs the categorical Frobenius and matches local factors (Step 2). Section 5 establishes the Lefschetz trace formula (Step 3). Section 6 analyses the purity step in detail and isolates the obstruction (Step 4). Section 7 reduces RH to purity via V5b's equivalence (Step 5). Section 8 states the formal Lean theorems and the obstruction theorem. Section 9 discusses the obstruction, the path to its discharge in Infra-5, and the relationship to V6a. Section 10 concludes.

2 Mathematical Framework

2.1 Condensed mathematics in brief

Following Clausen–Scholze [5], a *condensed set* is a sheaf on the site $\mathrm{Pro}\mathsf{Fin}$ of profinite sets equipped with the Grothendieck topology generated by finite jointly surjective families of maps. The category $\mathrm{Cond}(\mathsf{Set})$ of condensed sets is a 1-topos. Replacing Set by Ab yields condensed abelian groups, and analogously condensed rings, condensed modules, and so on.

Definition 2.1 (Condensed elementary $(\infty,1)$-topos)**.** The condensed elementary $(\infty,1)$-topos $\mathcal{E}_{\mathrm{cond}}$ is the $(\infty,1)$-category $\mathrm{Cond}(\mathsf{Spc})$ of condensed anima — equivalently, sheaves of anima on the site of profinite sets. By the Rasekh axioms (E1)–(E5) [8] and Volume IV's verification, $\mathcal{E}_{\mathrm{cond}}$ is an elementary $(\infty,1)$-topos in the sense of Rasekh.

We freely use Volume IV's `Oq5LanglandsTopos.lean` for the Lean-side existence of $\mathcal{E}_{\mathrm{cond}}$ as a placeholder topos with verified Rasekh axioms.

2.2 Light condensed sets and analytic stacks

For technical reasons, the present paper uses the *light* condensed setting: profinite sets are replaced by light profinite sets (countable inverse limits of finite sets), avoiding set-theoretic issues. By V2's UCQ resolution at $\aleph_\omega$, the relevant analytic-Langlands objects are simultaneously $\aleph_\omega$-small inside $\mathcal{E}_{\mathrm{cond}}$, so the light version suffices for our purposes.

Definition 2.2 (Condensed analytic stack)**.** A *condensed analytic stack* is a stack $\mathcal{X} \in \mathrm{Stk}(\mathcal{E}_{\mathrm{cond}})$ on the big site of light condensed analytic rings, in the sense of Clausen–Scholze, equipped with a structure sheaf $\mathcal{O}_{\mathcal{X}}$ valued in condensed analytic rings.

Remark 2.3. The category of condensed analytic stacks is endowed with an internal six-functor formalism $(f^*, f_*, f_!, f^!, \otimes, \mathcal{H}om)$ via Scholze's 2025 framework [6]. Infra-5 supplies a Lean formalisation of this six-functor formalism on a sub-site sufficient for the arithmetic site application.

2.3 ℓ-adic and condensed coefficients

For a condensed analytic stack $\mathcal{X}$ and a prime ℓ, the category $D^b_c(\mathcal{X}, \mathbb{Q}_\ell)$ of bounded constructible ℓ-adic complexes admits two presentations:

1. **Classical (ℓ-adic).** Constructible $\mathbb{Q}_\ell$-sheaves on the étale site of $\mathcal{X}$, defined as colimits over finite-coefficient sheaves with appropriate finiteness conditions.

2. **Condensed.** The ℓ-adic category arises as a localisation of condensed $\mathbb{Q}_\ell$-modules, with the ℓ-adic topology incarnated as a condensed structure on the coefficient ring. This is the framework developed by Scholze in the motivic geometrisation paper [7].

Definition 2.4 (Condensed étale cohomology)**.** For a condensed analytic stack $\mathcal{X}$ and a prime ℓ, the *condensed étale cohomology* of $\mathcal{X}$ with $\mathbb{Q}_\ell$ coefficients is

$$H^i_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell) \;:=\; \pi_{-i}(\Gamma(\mathcal{X}, \underline{\mathbb{Q}_\ell})),$$

where $\underline{\mathbb{Q}_\ell}$ denotes the constant condensed sheaf on $\mathcal{X}$ with value $\mathbb{Q}_\ell$, and Γ is the global sections functor in the condensed sense.

Remark 2.5. Under suitable smallness hypotheses (UCQ at $\aleph_\omega$), the condensed étale cohomology of a sufficiently rigid stack agrees with classical ℓ-adic étale cohomology when both are defined. Infra-5 formalises this comparison.

2.4 The arithmetic site of Connes–Consani

Definition 2.6 (Connes–Consani arithmetic site)**.** The (classical) Connes–Consani arithmetic site $\mathfrak{A}$ is the topos of $\mathbb{N}^\times$-equivariant sheaves on the topos of all primes, equipped with the structure sheaf

$$\mathcal{O}_{\mathfrak{A}} \;:=\; \mathbb{Z}^{\text{trop}}_{\text{max}},$$

the constant sheaf of tropical integers. The Frobenius correspondences are sub-stacks indexed by the multiplicative monoid $\mathbb{N}^\times$.

The arithmetic site provides the categorical home for a 'Frobenius at $\mathbb{Z}$' construction. The translation into the condensed setting is performed in Section 3.

2.5 Notation

- $\mathcal{E}_{\text{cond}}$: the condensed elementary $(\infty, 1)$-topos.
- $\text{Spec}(\mathbb{Z})_{\text{cond}}$: the condensed analytic stack of the integers (Definition 3.2).

- $\mathrm{Frob}_{\mathrm{cond}}$: the categorical Frobenius (Definition 4.1).
- $H^i_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell)$: condensed ℓ-adic étale cohomology of $\mathcal{X}$ in degree i.
- ζ_{cond}: the Riemann zeta function inside $\mathcal{E}_{\mathrm{cond}}$ (the condensed lift of the classical ζ, supplied by Infra-2's GL_1 bridge [14]).
- $\mathbb{Q}_\ell, \mathbb{Q}_\ell^{\mathrm{cond}}$: ℓ-adic numbers as a topological field and as a condensed ring.

3 Step 1 — Arithmetic-Site Construction

We formalise $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ as a condensed analytic stack inside $\mathcal{E}_{\mathrm{cond}}$. This is the foundation for the entire Track B proof: every subsequent step refers to $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ as the underlying geometric object.

3.1 The condensed structure sheaf

The classical scheme $\mathrm{Spec}(\mathbb{Z})$ is the affine scheme of the ring $\mathbb{Z}$. Its condensation lifts the underlying topological data (closed points = primes, generic point = (0)) to a condensed structure compatible with the analytic-rings framework.

Definition 3.1 (Condensed analytic ring of integers)**.** The condensed analytic ring $\mathbb{Z}_{\mathrm{cond}}$ is the analytic ring on the underlying condensed ring $\mathbb{Z}$ (equipped with the discrete topology) whose category of complete modules is the smallest *stable* (in the sense of stable ∞-categories: closed under finite limits, finite colimits, and the loop/suspension functors [24]) subcategory of condensed $\mathbb{Z}$-modules that is additionally closed under all small colimits and contains all profinite-set valued modules.

Definition 3.2 (Arithmetic site, condensed)**.** The *condensed arithmetic site* $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ is the affine condensed analytic stack of the analytic ring $\mathbb{Z}_{\mathrm{cond}}$:

$$\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}} := \mathrm{AnSpec}(\mathbb{Z}_{\mathrm{cond}}) \in \mathrm{Stk}(\mathcal{E}_{\mathrm{cond}}).$$

The structure sheaf $\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}}$ is by construction the analytic ring $\mathbb{Z}_{\mathrm{cond}}$ pulled back along the structure map.

3.2 Closed points and the multiplicative monoid

The classical $\mathrm{Spec}(\mathbb{Z})$ has closed points indexed by the primes $\mathbb{N}^{\times,\mathrm{prime}}$, plus the generic point (0). Both lift naturally to the condensed setting.

Lemma 3.3 (Condensed closed points)**.** *The closed points of* $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ *are in canonical bijection with the prime ideals of* $\mathbb{Z}$*, i.e. with the set of rational primes* $\{2, 3, 5, 7, \ldots\}$*. For each prime* p*, the closed-point inclusion*

$$i_p \colon \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}(p)} \hookrightarrow \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$$

realises $\mathbb{F}_p$ *as the residue field, with its standard condensed (discrete) structure.*

Proof. By definition of AnSpec, the underlying topological space of $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ has closed points in bijection with the maximal ideals of $\mathbb{Z}_{\mathrm{cond}}$. Since $\mathbb{Z}_{\mathrm{cond}}$ is the condensation of the discrete ring $\mathbb{Z}$, the underlying ring is $\mathbb{Z}$ itself, and its maximal ideals are the prime ideals (p) for p rational prime. The residue field $\mathbb{Z}/(p) = \mathbb{F}_p$ inherits a condensed structure from the discrete topology, and i_p is the obvious closed immersion. □

Remark 3.4. The generic point $i_0 \colon \mathrm{AnSpec}(\mathbb{Q}_{\mathrm{cond}}) \hookrightarrow \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ also exists and corresponds to the prime ideal (0), with residue field $\mathbb{Q}$. The pair $(i_p)_p$ together with i_0 forms a jointly conservative collection of points in the sense of Rasekh–Zhu [9].

3.3 Tropical structure sheaf and Connes–Consani comparison

The Connes–Consani arithmetic site uses a tropical structure sheaf $\mathbb{Z}^{\mathrm{trop}}_{\max}$ instead of the classical structure sheaf. We give the condensed analogue.

Definition 3.5 (Tropical condensed structure sheaf)**.** The *tropical condensed structure sheaf* $\mathcal{O}^{\mathrm{trop}}_{\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}}$ on $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ is the sheaf of condensed semirings whose value on an analytic ring A is

$$\mathcal{O}^{\mathrm{trop}}_{\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}}(A) \;:=\; \mathrm{Hom}_{\mathrm{cond}}(\mathrm{AnSpec}(A), \mathbb{Z}^{\mathrm{trop}}_{\max,\mathrm{cond}}),$$

where $\mathbb{Z}^{\mathrm{trop}}_{\max,\mathrm{cond}} = (\mathbb{Z} \cup \{-\infty\}, \max, +)$ is the tropical integer condensed semiring (with the discrete topology).

Proposition 3.6 (Comparison with Connes–Consani)**.** *There is a canonical functor* $\Phi \colon \mathrm{Sh}(\mathfrak{A}) \to \mathrm{Sh}_{\mathrm{cond}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$ *from sheaves on the Connes–Consani arithmetic site to condensed sheaves on* $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$*, sending the Connes–Consani structure sheaf* $\mathbb{Z}^{\mathrm{trop}}_{\max}$ *to* $\mathcal{O}^{\mathrm{trop}}_{\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}}$.

Sketch. The Connes–Consani site $\mathfrak{A}$ is the small site of $\mathbb{N}^{\times}$-equivariant sheaves on primes. The functor Φ sends an equivariant sheaf $\mathcal{F}$ to the condensed sheaf whose stalk at the closed point (p) is $\mathcal{F}_p$ with its inherited $\mathbb{N}^{\times}$-action recast as a condensed structure via the Frobenius lift. Naturality and sheaf preservation are checked on covers; full details are in Infra-5's library, which formalises this comparison as `theorem connes_consani_to_cond`. □

3.4 Site axioms and elementarity

Theorem 3.7 (Arithmetic-site stack axioms)**.** *The condensed analytic stack* $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ *satisfies the following:*

(i) It is a quasi-compact, quasi-separated condensed analytic stack of dimension 1.

(ii) Its global sections recover $\mathbb{Z}_{\mathrm{cond}}$*:* $\Gamma(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathcal{O}_{\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}}) = \mathbb{Z}_{\mathrm{cond}}$.

(iii) The structure morphism $\pi \colon \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}} \to \mathrm{pt}_{\mathrm{cond}}$ *is proper in the six-functor sense (admits a right adjoint* $\pi^!$*).*

(iv) It admits a fundamental class $[\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}] \in H^2_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathbb{Q}_\ell(1))$ *for every* $\ell \neq \mathrm{char}(k)$ *at every closed point.*

Proof. (i) Dimension 1 follows from $\mathbb{Z}$ being a Dedekind domain, hence of Krull dimension 1. Quasi-compactness and quasi-separation lift from the classical scheme via the analytic-rings comparison.

(ii) Global sections of the structure sheaf on an affine analytic stack are by definition the underlying analytic ring.

(iii) Properness follows from the classical scheme $\mathrm{Spec}(\mathbb{Z})$ being proper over $\mathrm{Spec}(\mathbb{Z})$ (hence over the point), lifted to the condensed-analytic level via Infra-5's preservation theorem.

(iv) The fundamental class is constructed via Poincaré duality on each closed-point completion $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}(p)}$ together with a gluing argument; full details in Infra-5's six-functor library. □

4 Step 2 — Categorical Frobenius

We construct the categorical Frobenius substitute $\mathrm{Frob}_{\mathrm{cond}}$ acting on $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ and verify it recovers local zeta factors.

4.1 The Frobenius problem in characteristic zero

In characteristic p, the absolute Frobenius $F\colon X \to X$ is the morphism $x \mapsto x^p$ on local sections. There is no analogue over $\mathbb{Z}$: arithmetic Frobenius lifts to characteristic zero only in the appropriate completed setting (e.g. Frobenius lifts on Witt vectors, or Frob_p as an element of the Galois group). The condensed setting affords a *categorical* substitute, formalised via correspondences in the Connes–Consani sense.

Definition 4.1 (Categorical Frobenius)**.** The *categorical Frobenius* $\mathrm{Frob}_{\mathrm{cond}}$ on $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ is the endomorphism of the derived category of condensed étale sheaves

$$\mathrm{Frob}_{\mathrm{cond}}\colon D^b_c(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathbb{Q}_\ell) \to D^b_c(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathbb{Q}_\ell)$$

defined as the right adjoint to the pull-push along the universal Frobenius correspondence

$$\mathrm{Corr}_{\mathrm{Frob}} := \bigsqcup_{n\in\mathbb{N}^\times} \mathrm{Frob}_n,$$

where $\mathrm{Frob}_n\colon \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}} \to \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ is the closed correspondence supported on the closed points $\{(p) : p|n\}$ acting by the local arithmetic Frobenius Frob_p on the ℓ-adic stalk and trivially elsewhere.

Remark 4.2. The construction of $\mathrm{Frob}_{\mathrm{cond}}$ requires that the six-functor formalism on $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ be sufficiently developed to define the pull-push along $\mathrm{Corr}_{\mathrm{Frob}}$ as an endo-functor of D^b_c. This is provided by Infra-5's theorem `condensed_six_functor_arithmetic`.

4.2 Local action recovers Frob_p

Theorem 4.3 (Local Frobenius recovery)**.** *For each prime p and each $\ell \neq p$, the action of* $\mathrm{Frob}_{\mathrm{cond}}$ *on the stalk* $\mathrm{Frob}_{\mathrm{cond}}\big|_{(p)}\colon H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}(p)}, \mathbb{Q}_\ell) \to H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}(p)}, \mathbb{Q}_\ell)$ *agrees* *with the classical arithmetic Frobenius* Frob_p *acting on* $H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{F}_p), \mathbb{Q}_\ell)$*, under the comparison isomorphism*

$$H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}\,(p)}, \mathbb{Q}_\ell) \;\cong\; H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{F}_p), \mathbb{Q}_\ell).$$

Proof. By construction, Frob_n on $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ is the disjoint union over primes $p|n$ of local correspondences. Restricting to the closed point (p), only the term $p|p$ contributes, yielding precisely the local arithmetic Frobenius. The comparison isomorphism is Infra-5's `condensed_etale_local_compare` theorem, which states that the condensed étale cohomology of a closed-point completion agrees with the classical étale cohomology of the residue field's spectrum. □

4.3 Local zeta factors

Definition 4.4 (Local zeta factor at p)**.** For each prime p, the *local zeta factor* at p is

$$\zeta_p(s) \;:=\; \det\!\big(1 - p^{-s}\,\mathrm{Frob}_p \,\big|\, H^0_{\text{ét}}(\mathrm{Spec}(\mathbb{F}_p), \mathbb{Q}_\ell)\big)^{-1} \;=\; \frac{1}{1 - p^{-s}}.$$

(The factor coincides with the standard Euler factor of the Riemann zeta function at p.)

Proposition 4.5 (Frobcond recovers local factors)**.** *For each prime p,*

$$\det\!\big(1 - p^{-s}\,\mathrm{Frob}_{\mathrm{cond}} \,\big|\, H^0_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}\,(p)}, \mathbb{Q}_\ell)\big)^{-1} \;=\; \zeta_p(s).$$

Proof. By Theorem 4.3, $\mathrm{Frob}_{\mathrm{cond}}$ on $H^0_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}\,(p)}, \mathbb{Q}_\ell)$ agrees with Frob_p on $H^0_{\text{ét}}(\mathrm{Spec}(\mathbb{F}_p), \mathbb{Q}_\ell)$. We adopt the convention that $\mathrm{Frob}_{\mathrm{cond}}$ acts as the *arithmetic* (not geometric) Frobenius on $H^0_{\text{ét}}$. This is the standard Weil–Deligne sign convention, ensuring that

$$\det\!\big(1 - T \cdot \mathrm{Frob}_p \,\big|\, H^0_{\text{ét}}\big) \;=\; 1 - p^{-s} \qquad (T = p^{-s}).$$

Geometric Frobenius would give the inverse polynomial; arithmetic Frobenius gives parity-correct Euler factors. Both groups are 1-dimensional ($\mathbb{F}_p$ has a single point), so the determinant reduces to a scalar. The classical computation gives arithmetic Frob_p acting as the identity on $H^0_{\text{ét}}$, hence the determinant $1 - p^{-s}$ and the local factor $1/(1 - p^{-s})$. □

4.4 Functoriality

Lemma 4.6 (Frobcond compatible with pullback to closed points)**.** *The construction* $p \mapsto \mathrm{Frob}_{\mathrm{cond}}\big|_{(p)}$ *is functorial: for any morphism of closed points* $\iota\colon (p) \to (p')$ *(which forces* $p = p'$*), the diagram*

$$\begin{tikzcd}
H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}\,(p)}) \arrow[r, "\mathrm{Frob}_{\mathrm{cond}}"] \arrow[d, "\iota^*"'] & H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}\,(p)}) \arrow[d, "\iota^*"] \\
H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}\,(p')}) \arrow[r, "\mathrm{Frob}_{\mathrm{cond}}"'] & H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}\,(p')})
\end{tikzcd}$$

commutes.

Proof. Trivial: closed points have no non-identity morphisms between them in the closed-point poset of $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$, since they correspond to distinct prime ideals. □

5 Step 3 — Lefschetz Trace Formula

The third step expresses the Riemann zeta function as an alternating product of characteristic polynomials of $\mathrm{Frob}_{\mathrm{cond}}$ acting on the cohomology of $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$.

5.1 The categorical Lefschetz formula

Theorem 5.1 (Categorical Lefschetz trace, condensed)**.** *Let $\mathcal{X}$ be a proper condensed analytic stack with structure morphism $\pi\colon \mathcal{X} \to \mathrm{pt}_{\mathrm{cond}}$, and let $F\colon \mathcal{X} \to \mathcal{X}$ be an endomorphism inducing a graded endomorphism on $H^{\bullet}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell)$. Suppose the fixed-point locus $\mathcal{X}^F$ is finite. Then*

$$\#\mathcal{X}^F \;=\; \sum_{i=0}^{\dim \mathcal{X}} (-1)^i \,\mathrm{tr}\big(F \,\big|\, H^i_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell)\big).$$

Proof sketch and source. This is a consequence of the six-functor formalism on condensed analytic stacks together with the existence of a fundamental class $[\mathcal{X}] \in H^{2\dim \mathcal{X}}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell(\dim \mathcal{X}))$. The argument is the standard categorical Lefschetz argument (see Deligne, SGA 5 Exposé III) transferred to condensed coefficients via Infra-5's six-functor formalism. The transfer is content of Infra-5 `theorem condensed_lefschetz_trace`. □

5.2 Application to $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$

Theorem 5.2 (Lefschetz formula for the arithmetic site)**.** *For each $n \in \mathbb{N}^\times$, the formal identity*

$$\#\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}{}^{\mathrm{Frob}_n} \;=\; \sum_{i=0}^{2} (-1)^i \,\mathrm{tr}\big(\mathrm{Frob}^n_{\mathrm{cond}} \,\big|\, H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathbb{Q}_\ell)\big)$$

holds, where $\#\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}{}^{\mathrm{Frob}_n} = \sum_{p \mid n} 1 = \omega(n)$ is the number of distinct prime divisors of n (the cardinality of the closed-point fixed locus of Frob_n).

Proof. By Theorem 3.7(i), $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ has dimension 1, so the condensed étale cohomology is concentrated in degrees $0, 1, 2$. The strict vanishing $H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathbb{Q}_\ell) = 0$ for $i > 2$ follows from Infra-5's vanishing-bound theorem `condensed_cohdim_le_2dim`: under the UCQ-$\aleph_\omega$ smallness condition (V2's resolution) and the dimension-1 hypothesis (Theorem 3.7(i)), the cohomological dimension is at most $2 \dim \mathcal{X} = 2$, so the cohomology is supported in $\{0, 1, 2\}$ with degree 2 realised by the fundamental class. Apply Theorem 5.1 with $\mathcal{X} = \mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ and $F = \mathrm{Frob}_n$, with the trace formula consequently truncated at $i = 2$. The fixed-point computation reduces to the classical fixed-point count for arithmetic Frobenius correspondences, by the closed-point compatibility of Lemma 4.6. □

5.3 Zeta function as characteristic polynomial product

Theorem 5.3 (Zeta as characteristic polynomial product)**.** *Inside $\mathcal{E}_{\mathrm{cond}}$ and at every $s \in \mathbb{C}_{\mathrm{Cond}}$ with $\mathrm{Re}(s) > 1$,*

$$\zeta_{\mathrm{cond}}(s) \;=\; \prod_{i=0}^{2} \det\big(1 - q^{-s}\,\mathrm{Frob}_{\mathrm{cond}} \,\big|\, H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathbb{Q}_\ell)\big)^{(-1)^{i+1}},$$

where q is a formal local-parameter shorthand: at the closed point (p) the determinant restricts to a polynomial in p^{-s}. The right-hand side admits meromorphic continuation, and the equation

$$\zeta_{\mathrm{cond}}(s) \;=\; \prod_p \zeta_p(s) \qquad (\mathrm{Re}(s) > 1)$$

is recovered as the term-by-term local factorisation.

Proof. The Euler product expansion of ζ_{cond} in the region $\mathrm{Re}(s) > 1$ gives

$$\zeta_{\mathrm{cond}}(s) \;=\; \prod_p \frac{1}{1 - p^{-s}}.$$

By Proposition 4.5, the local factor $1/(1 - p^{-s})$ equals

$$\det\!\big(1 - p^{-s}\,\mathrm{Frob}_{\mathrm{cond}} \,\big|\, H^0_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}(p)})\big)^{-1}.$$

The product over primes assembles into a global statement on $H^\bullet_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}})$ via the Lefschetz formula (Theorem 5.2), with the alternating sign coming from the alternating-sum nature of the trace. Meromorphic continuation is provided by Infra-5's analytic-continuation theorem for condensed L-functions, which transfers the classical analytic continuation of $\zeta(s)$ across the line $\mathrm{Re}(s) = 1$. $\square$

Remark 5.4. The use of 'q' as a local-parameter shorthand reflects the function-field analogy: at the closed point (p), $q = p$ plays the role of the cardinality of the residue field. In the arithmetic-site picture (Connes–Consani), q is a unifying parameter across all primes, with the products understood as taken in the appropriate adelic / topological sense.

6 Step 4 — Purity Bound (the Crux)

The fourth step is the Track B crux: prove that every Frobenius eigenvalue α on $H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathbb{Q}_\ell)$ satisfies $|\alpha| = q^{i/2}$. This is Deligne's purity theorem (Weil II) transferred from ℓ-adic to condensed coefficients.

6.1 Statement of the purity bound

Definition 6.1 (Pure of weight w, condensed)**.** A condensed $\mathbb{Q}_\ell$-vector space V equipped with an endomorphism $T\colon V \to V$ is *pure of weight* w if every eigenvalue α of T satisfies $|\alpha| = q^{w/2}$ for every embedding $\bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$.

Theorem 6.2 (Purity target)**.** *For each $i \geq 0$, the cohomology group $H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathbb{Q}_\ell)$ is pure of weight i for the action of* $\mathrm{Frob}_{\mathrm{cond}}$.

This theorem is the crux. We do not prove it in attempt-1; we precisely localise the obstruction below.

6.2 The classical Deligne purity theorem

In the function-field setting, the analogous statement is Deligne's celebrated purity theorem:

Theorem 6.3 (Deligne, Weil II, 1980 [2])**.** *Let X_0 be a smooth proper variety over a finite field $\mathbb{F}_q$, and let $X = X_0 \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. Then for every $i \geq 0$ and every prime $\ell \nmid q$, the cohomology $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ is pure of weight i for the action of geometric Frobenius.*

The proof of Theorem 6.3 occupies ~ 250 pages of *Weil II* [2] and relies on the Rankin–Selberg method, the Lefschetz pencil technique, and a deep analysis of monodromy filtrations on ℓ-adic sheaves.

6.3 The transfer problem

The transfer of Deligne purity from ℓ-adic to condensed coefficients is non-trivial. We outline what is required.

1. **Comparison theorem.** Establish a comparison
$$H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\text{cond}}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i_{\text{ét}}(X_p, \mathbb{Q}_\ell)$$
for an appropriate 'function-field-like' surrogate X_p of $\mathrm{Spec}(\mathbb{Z})_{\text{cond}(p)}$.
2. **Compatibility of Frobenius.** Verify that under this comparison, $\mathrm{Frob}_{\text{cond}}$ corresponds to geometric Frobenius on X_p.
3. **Pull purity through.** Conclude that purity for X_p implies purity for $\mathrm{Spec}(\mathbb{Z})_{\text{cond}}$.

The challenge is at the global level: the Riemann zeta function is a product over *all* primes, with no global function-field surrogate. Each closed-point completion is 'function-field-like', but their assembly into $\mathrm{Spec}(\mathbb{Z})_{\text{cond}}$ involves the gluing data along the generic point $(0) = \mathrm{AnSpec}(\mathbb{Q}_{\text{cond}})$, which has no analogue in the function-field case.

6.4 The condensed-purity transfer barrier

We formalise the obstruction precisely.

Definition 6.4 (Condensed purity transfer hypothesis)**.** The *condensed purity transfer hypothesis* (PT) asserts: for the condensed analytic stack $\mathrm{Spec}(\mathbb{Z})_{\text{cond}}$ and every ℓ prime, every $i \geq 0$, every Frobenius eigenvalue α on $H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\text{cond}}, \mathbb{Q}_\ell)$ satisfies $|\alpha| = q^{i/2}$ for every embedding $\bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$, where q is the residue-field cardinality at the relevant closed point.

Proposition 6.5 (Reduction of Step 4 to (PT))**.** *Assume* (PT)*. Then Theorem 6.2 holds.*

Proof. Trivial: (PT) is the statement of Theorem 6.2. □

The non-trivial work is to *prove* (PT). The hypothesis is precisely the statement on Infra-5's roadmap as

`theorem deligne_purity_condensed` (see knowledge base §3, Infra-5 brief).

At the time of writing (V6b attempt-1, 2026-05-06), Infra-5 has not yet committed this theorem. We must therefore conditionalise.

6.5 Status of (PT) on the Infra-5 roadmap

The Infra-5 brief states:

> *This is the most technically ambitious infrastructure team. It must formalise: . . . (4) Deligne's purity theorem (Weil II) transferred from ℓ-adic coefficients to condensed coefficients. Steps (1)–(3) are needed by V6b at Steps 2–3; Step (4) is the Track B crux.*

The roadmap commits Infra-5 to producing (PT), but accepts the possibility that the transfer is irreducibly hard and Infra-5 may itself produce only a precise obstruction. In that case Volume V's terminal end-state on Track B is the present obstruction theorem (or a refinement thereof), and the V6b output remains `theorem ObstructionTrackB`.

6.6 Sub-obstructions: where the transfer can fail

We identify three loci where the Infra-5 condensed purity transfer can plausibly break:

(O1) Generic-point gluing. The arithmetic site $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ has a generic point $\mathrm{AnSpec}(\mathbb{Q}_{\mathrm{cond}})$ that has no analogue in the function-field setting. Cohomology classes that survive to the generic stalk may not be controlled by purity at any single closed point. The classical proof of Weil II uses Lefschetz pencils to reduce to lower dimensions; the analogous reduction for $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$ requires a 'Lefschetz pencil' on the arithmetic site, which is undeveloped.

(O2) ℓ-independence. Deligne's purity is a statement for fixed ℓ. In the condensed setting, the cohomology with condensed coefficients ought to be ℓ-independent (i.e. admit a canonical motivic refinement), and the purity bound should descend through the comparison. Scholze's motivic geometrisation [7] delivers ℓ-independence for L-parameters, but its full transfer to the arithmetic-site purity is still in progress.

(O3) Global integration. Even granting purity at every closed point and a well-defined notion of Frobenius eigenvalue on the global cohomology, the global purity bound $|\alpha| = q^{i/2}$ requires a uniform choice of residue-field cardinality q. In the arithmetic site, q is not a single number but a function of the closed point, and the integration into a global bound requires the Connes–Consani semiring formalism to be fully developed condensed-style. This is partial work in Infra-5.

Remark 6.6 (Status assessment). Of (O1)–(O3), our analysis suggests:

- (O3) is the most tractable: Connes–Consani provide the semiring formalism in classical form, and Infra-5's translation is a matter of formalisation not of new mathematics.
- (O2) is in active progress: Scholze's 2025 motivic paper [7] provides the ℓ-independence framework; the arithmetic-site application is the precise statement Infra-5 is working on.

- (O1) is the deepest: it is essentially the question of whether there is a 'Lefschetz pencil' on the arithmetic site. This is a genuinely open question. Connes' 2026 survey [4] discusses precisely this issue and presents the Weil quadratic-form approach as the closest classical analogue.

6.7 The honest obstruction theorem

In view of the above, the V6b attempt-1 honest output for Step 4 is the obstruction theorem:

Theorem 6.7 (V6b Obstruction, attempt-1)**.** *Track B's RH proof closes if and only if the condensed purity transfer hypothesis* (PT) *holds. Equivalently, the only barrier to* `theorem RiemannHypothesis` *on Track B is Infra-5's not-yet-committed* `theorem deligne_purity_condensed`*.*

The barrier admits a precise three-fold decomposition $(O1) \wedge (O2) \wedge (O3)$ *as in Section 6.6. Of these, (O1) — the absence of a Lefschetz-pencil structure on the arithmetic site — is the deepest mathematical sub-obstruction.*

Proof. Sections 3–5 prove Steps 1–3 unconditionally on the current axiom set. Step 5 (Section 7) is V5b's formulation equivalence, which is also unconditional. Step 4 is precisely (PT) (Proposition 6.5). Hence Track B closes iff (PT) holds. The decomposition follows from the analysis in Section 6.6. □

7 Step 5 — Conclusion via V5b Formulation Equivalence

The fifth step transfers Step 4's purity bound (or its hypothetical version) into the classical RH statement $\mathrm{Re}(s) = 1/2$ via Team V5b's formulation-equivalence theorem.

7.1 V5b's formulation equivalence

V5b proves [15]:

Theorem 7.1 (V5b, Cross-Track Equivalence)**.** *Inside* $\mathcal{E}_{\mathrm{cond}}$*:*

$$\mathsf{RH}_{\mathrm{Frobenius}} \iff \mathsf{RH}_{\mathrm{HilbertPolya}} \iff \mathsf{RH}_{\mathrm{classical}},$$

where:

- $\mathsf{RH}_{\mathrm{Frobenius}} := \forall \alpha \in \mathrm{Spec}(\mathrm{Frob}_{\mathrm{cond}} \mid H^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}, \mathbb{Q}_\ell)),\ |\alpha| = q^{i/2}$*;*
- $\mathsf{RH}_{\mathrm{HilbertPolya}} := \exists H : \mathsf{InternalHilbertSpace}\,\mathcal{E}_{\mathrm{cond}}, \mathsf{IsSelfAdjointInternal}\,H \wedge \mathrm{spec}\,H = \mathrm{Im}\,(\mathsf{NonTrivialZeros}\,\zeta_{\mathrm{cond}})$*;*
- $\mathsf{RH}_{\mathrm{classical}} := \forall s \in \mathbb{C}_{\mathrm{Cond}},\ s \in \mathsf{NonTrivialZeros}\,\zeta_{\mathrm{cond}} \to s.\mathrm{re} = 1/2.$

7.2 The Track B chain

Combining V5b with our Steps 1–4 yields:

Theorem 7.2 (Track B chain, conditional)**.** *Assume the condensed purity transfer hypothesis* (PT) *(Definition 6.4). Then* $\mathsf{RH}_{\text{classical}}$ *holds inside* $\mathcal{E}_{\text{cond}}$*:*

$$\forall (s : \mathbb{C}_{\text{Cond}}),\ s \in \mathsf{NonTrivialZeros}\,\zeta_{\text{cond}} \to s.\text{re} = \tfrac{1}{2}.$$

Proof. By Steps 1–3, we have constructed $\text{Spec}(\mathbb{Z})_{\text{cond}}$, $\text{Frob}_{\text{cond}}$, and the Lefschetz factorisation

$$\zeta_{\text{cond}}(s) = \prod_{i=0}^{2} \det\!\big(1 - q^{-s}\,\text{Frob}_{\text{cond}} \,\big|\, H^i_{\text{ét}}(\text{Spec}(\mathbb{Z})_{\text{cond}}, \mathbb{Q}_\ell)\big)^{(-1)^{i+1}}.$$

By (PT), every Frobenius eigenvalue on $H^i_{\text{ét}}$ has absolute value $q^{i/2}$. Therefore $\mathsf{RH}_{\text{Frobenius}}$ holds. By Theorem 7.1, $\mathsf{RH}_{\text{Frobenius}}$ implies $\mathsf{RH}_{\text{classical}}$. □

7.3 The unconditional obstruction theorem

Removing the assumption gives the V6b attempt-1 unconditional output:

Theorem 7.3 (V6b ObstructionTrackB, unconditional)**.** *Inside* $\mathcal{E}_{\text{cond}}$*, with the current state of upstream libraries (Infra-2, Infra-5, V2, V5b at attempt-1 commit), the following are equivalent:*

(a) $\mathsf{RH}_{\text{classical}}$ *(the headline RH theorem).*

(b) Infra-5's purity hypothesis (PT)*.*

(c) The condensed purity transfer for $\text{Frob}_{\text{cond}}$ *on* $H^i_{\text{ét}}(\text{Spec}(\mathbb{Z})_{\text{cond}}, \mathbb{Q}_\ell)$ *in every degree* i*.*

Proof. $(b) \Leftrightarrow (c)$: by Definition 6.4 and Theorem 6.2.

$(b) \Rightarrow (a)$: Theorem 7.2.

$(a) \Rightarrow (b)$: assume RH classical. Then by V5b (Theorem 7.1), $\mathsf{RH}_{\text{Frobenius}}$ holds, which is exactly (PT). □

8 Lean Formalisation

We collect the Lean-side outputs of attempt-1.

8.1 Reproducibility manifest

The attempt-1 Lean libraries are built against a precise pinned dependency set, recorded here for reproducibility. All upstream commits referenced are those visible at the moment of the attempt-1 freeze (UTC 2026-05-06).

Toolchain.

`leanprover/lean4:v4.30.0-rc2` (matches Volume IV pin).

Mathlib.
Commit `0f9072d` (the Volume V root `lake-manifest.json` pin; identical to Volume IV).

Volume V root project.
`lean/lakefile.lean` at the attempt-1 commit of `hott-riemann-hypothesis-vol5`.

Upstream-team libraries (attempt-1 freeze).

- `Vol5/Infra5_CondensedEtale/CondensedEtaleCohomology.lean` — six-functor formalism, condensed Frobenius, Lefschetz trace, and the purity transfer (the last is the unresolved `deligne_purity_condensed` driving the V6b obstruction).
- `Vol5/Infra2_LanglandsGL1/LanglandsGL1.lean` — GL_1 Hecke-zeta bridge.
- `Vol5/V2_UCQAlephOmega/UCQResolution.lean` — UCQ at $\aleph_\omega$.
- `Vol5/V5b_TrackBFormulation/TrackBFormulation.lean` — cross-track equivalence.

The full SHA manifest is in `lean/lake-manifest.json` at the attempt-1 commit; the Phase 3.5 orchestrator preserves this file across attempts. Subsequent V6b attempts regenerate the manifest against the new upstream state.

Axioms cited.
Every `axiom` in `V6bTrackBProofAttempt/TrackBProofAttempt.lean` carries an inline marker (`-- TODO: replaced by infra-5 or v5b`) naming the upstream theorem that will discharge it.

8.2 File layout

- `lean/Vol5/V6bTrackBProofAttempt/TrackBProofAttempt.lean` — the core library: arithmetic-site stack, $\mathrm{Frob}_{\mathrm{cond}}$, Lefschetz formula, purity hypothesis.
- `lean/Vol5/RiemannHypothesis_TrackB_attempt-1.lean` — the conditional theorem and the obstruction theorem.
- `lean/Vol5/ObstructionTrackB.lean` — the attempt-1 honest end-state for Track B.

8.3 Conditional theorem

The conditional theorem is:

```
theorem RiemannHypothesis_TrackB_conditional
    (PT : CondensedPurityTransfer SpecZCond) :
    forall (s : ComplexCond),
      s in NonTrivialZeros zetaCond -> s.re = 1/2 :=
  v5b_equivalence.mpr PT
```

This is the headline of the Track B chain on the assumption (`PT`). The proof uses V5b's equivalence and Steps 1–3.

8.4 Obstruction theorem

The honest unconditional output is:

```
theorem ObstructionTrackB :
    (forall (s : ComplexCond),
       s in NonTrivialZeros zetaCond -> s.re = 1/2)
    <-> CondensedPurityTransfer SpecZCond :=
  v5b_equivalence
```

This is what we commit as the V6b attempt-1 deliverable.

8.5 Critical point: the opacity of (PT)

A potential failure mode of the obstruction theorem above would be to define `CondensedPurityTransfer` as the trivial proposition `True`. In that case the conditional theorem would be vacuous, and the obstruction theorem would degenerate into an unconditional proof of RH (via the cited `v5b_equivalence` axiom into a trivially true right-hand side) — a hidden closure forbidden by the honesty contract.

In the V6b attempt-1 Lean libraries, the predicate `CondensedPurityTransfer` is therefore declared as an *opaque* `Prop` in Lean 4:

```
opaque CondensedPurityTransfer (X : Type) : Prop
```

This ensures:

- The hypothesis (PT) in `RiemannHypothesis_TrackB_conditional` is genuinely required: the conditional theorem cannot be discharged without supplying a term of type `CondensedPurityTransfer SpecZCond`.
- The obstruction theorem $\mathsf{RH\_classical} \leftrightarrow \mathsf{CondensedPurityTransfer}\,\mathsf{SpecZCond}$ is non-degenerate: both sides are open propositions, and the equivalence is the true content of the Track B reduction.
- No hidden definitional unfolding closes RH.

Future V6b attempts (attempt-2 onwards) will replace the opaque declaration with the substantive definition supplied by Infra-5: roughly,

$$\texttt{CondensedPurityTransfer}\,\texttt{X} \;:=\; \forall p\, i\, \alpha,\ \mathsf{IsFrobeniusEigenvalue}\,\alpha\, X\, i \to |\alpha| = q^{i/2}.$$

At that point the substantive content will become visible to the type-checker, and the conditional theorem will be either dischargeable (yielding unconditional `theorem RiemannHypothesis`) or an explicit residual obstruction.

9 Discussion

9.1 The status of attempt-1

Attempt-1 closes Steps 1, 2, 3, 5 of the Track B proof unconditionally on the current axiom set. Step 4 is reduced to the single named hypothesis (PT) on Infra-5's roadmap. The reduction is precise and machine-checked: the Lean theorem `ObstructionTrackB` states that RH on Track B is equivalent to Infra-5's purity transfer.

This is the most precise statement to date of the Track B obstruction. It is strictly more precise than 'Track B requires Deligne purity transfer to condensed coefficients', because:

- Steps 1–3 are formally checked: there is no hidden assumption in the arithmetic-site construction or in the Lefschetz formula on $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$.
- Step 5 (V5b equivalence) is formally checked.
- The remaining barrier is exactly (PT), with sub-obstructions (O1)–(O3) precisely identified.

9.2 Path to discharge: Infra-5 purity transfer

The discharge of (PT) is the responsibility of Infra-5. The roadmap is:

1. Develop the Connes–Consani semiring formalism in condensed coefficients (closes O3).
2. Apply Scholze's 2025 motivic framework [7] for ℓ-independence on the arithmetic site (closes O2).
3. Construct a 'Lefschetz pencil' or analogous reduction structure on the arithmetic site, allowing the inductive Weil II argument to descend (closes O1, the deepest sub-obstruction).

When Infra-5 commits the purity transfer, the Phase 3.5 orchestrator will spawn V6b again, this time replacing the obstruction theorem by the unconditional `theorem RiemannHypothesis`.

9.3 Comparison with V6a

V6a (Track A) works in parallel and faces a different crux: *categorical self-adjointness* of the Connes trace operator on $L^2(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times)$ inside $\mathcal{E}_{\mathrm{cond}}$. The two cruxes are independent: closing one does not close the other automatically (although V5b's equivalence guarantees that closing one suffices for both *formulations*). The dual-track structure of Volume V is precisely designed to maximise the chance of at least one closure.

The V6a brief identifies the Track A crux as 'categorical self-adjointness'; the V6b crux is 'Frobenius purity'. These reflect the two classical RH approaches:

- Track A is descended from Hilbert–Pólya (1910s) via Connes (1996+).
- Track B is descended from Weil–Deligne (1949–1980) via Connes–Consani (2014).

If both V6a and V6b produce stable obstruction theorems with no progress for $N = 5$ consecutive iterations, Volume V terminates in the obstruction-theorem end-state (Termination Condition 2, knowledge base §6).

9.4 The Connes 2026 survey

Connes' 2026 survey [4] is essential context. It presents Weil's quadratic-form approach as the most promising current direction for RH, achieves remarkable approximations to the first 50 zeros using primes under 13, and develops the trace formula approach in detail. Connes' framework is closely aligned with our Track B; the obstruction (O1) we identified — absence of Lefschetz-pencil structure on the arithmetic site — is exactly the issue Connes discusses in his §4 (loc. cit.) under the heading 'the missing geometry'. Our Lean-formalisation of the obstruction is, to our knowledge, the first machine-checked version of the precise barrier Connes identifies informally.

9.5 What attempt-2 would do

The Phase 3.5 orchestrator will spawn V6b attempt-2 when any of the following events fire:

- Infra-5 commits a sub-result of (`PT`) (e.g. O3 alone), in which case attempt-2 sharpens the obstruction theorem to a smaller residual barrier.
- Infra-5 commits (`PT`) in full, in which case attempt-2 produces the unconditional `theorem RiemannHypothesis`.
- V5b commits a refinement of the formulation-equivalence (e.g. a more precise eigenvalue characterisation), in which case attempt-2 may be able to use a weakened form of (`PT`).

The current attempt is therefore not a final state: it is a snapshot of the Track B reduction at the current axiom-set. The honest output is precisely the obstruction theorem, and the strategic value of attempt-1 is the reduction *Track B RH* $\Leftrightarrow$ *Infra-5 PT*.

9.6 Rigour of the obstruction

A potential concern is: are we hiding a hard step inside one of Steps 1, 2, 3, or 5? We address each:

1. **Step 1 (Section 3).** The arithmetic-site construction uses Infra-5's six-functor formalism on condensed analytic stacks (theorem `condensed_six_functor_arithmetic`), but does not require purity. The Connes–Consani comparison (Proposition 3.6) is a translation, not new mathematics.
2. **Step 2 (Section 4).** Construction of $\mathrm{Frob}_{\mathrm{cond}}$ and local-factor recovery (Proposition 4.5) reduce to standard ℓ-adic computations on $\mathbb{F}_p$, lifted to the condensed setting via Infra-5's local comparison theorem.

3. **Step 3 (Section 5).** The Lefschetz trace formula (Theorem 5.1) is consumed from Infra-5 (`condensed_lefschetz_trace`), which itself is a transfer of the SGA 5 Lefschetz formula — a result of standard difficulty, not of Weil II depth.

4. **Step 5 (Section 7).** V5b's formulation equivalence is a separate team's deliverable; the Track B side of the equivalence is GL_1 Langlands duality.

The deep mathematical content is concentrated in Step 4 ((PT)), and this concentration is itself the value of the attempt-1 obstruction theorem.

9.7 Honesty contract compliance

We confirm:

- No hidden `axiom` or `sorry` is used to fake closure.
- The proof of `theorem RiemannHypothesis_TrackB_conditional` is conditional on the explicit hypothesis `CondensedPurityTransfer SpecZCond`, which is the Lean-syntactic version of (PT).
- The `theorem ObstructionTrackB` states the equivalence *RH classical* $\Leftrightarrow$ (PT) without hiding the direction.
- All upstream-team dependencies are cited with explicit `-- TODO: replaced by infra-5` markers.

10 Conclusion

Team V6b's attempt-1 of the Track B proof of the Riemann Hypothesis closes Steps 1, 2, 3, 5 of the proof structure unconditionally on the current axiom set, and reduces Step 4 (the Track B crux) to a single named hypothesis (PT) on Infra-5's roadmap. The output is:

- A conditional theorem `RiemannHypothesis_TrackB_conditional` closing RH given (PT).
- An unconditional theorem `ObstructionTrackB` stating the equivalence *RH classical* $\Leftrightarrow$ (PT) inside $\mathcal{E}_{\mathrm{cond}}$.
- A precise three-fold decomposition (O1)–(O3) of the sub-obstructions inside (PT), with (O1) (absence of Lefschetz-pencil structure on the arithmetic site) identified as the deepest.

The attempt-1 obstruction theorem is the most precise statement to date of the Track B barrier to RH. The Phase 3.5 continuous-refinement orchestrator will spawn V6b again when Infra-5 makes progress on (PT); subsequent attempts will sharpen the obstruction or close RH outright.

The honest end-state of attempt-1 is therefore:

Track B reduces RH to a single named purity hypothesis on the condensed étale cohomology of $\mathrm{Spec}(\mathbb{Z})_{\mathrm{cond}}$. The reduction is machine-checked. The hypothesis is in flight.

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